

Ramsey numbers of sparse digraphs

Yuval Wigderson (Stanford)

Joint with Jacob Fox and Xiaoyu He

April 16, 2021

Warmup: Hamiltonian paths in tournaments

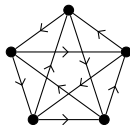
Theorem (Rédei 1934)

Every tournament contains a Hamiltonian path.

Tournament = complete directed graph (every pair of vertices connected by a directed edge)

Questions and results about Hamiltonian paths in tournaments abound!

What structures must appear in every N -vertex tournament?



Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N -vertex tournament contains a copy of H .

Rédei's theorem $\iff \vec{r}(P_n) = n$, where P_n = directed n -vertex path.

Directed and undirected Ramsey numbers

Definition

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every **two-edge-coloring** of K_N contains a **monochromatic** copy of H .

For a complete graph K_n ,

$$2^{n/2} \leq r(K_n) \leq 2^{2n}.$$

The upper bound implies that $r(H)$ exists for all H .

If H has εn^2 edges, then

$$r(H) \geq 2^{\varepsilon n}.$$

So the Ramsey number is exponential if H is **dense**.

For the rest of the talk, we'll focus on **sparse** (di)graphs.

Definition

The *Ramsey number* $\vec{r}(H)$ of a digraph H is the minimum N such that every N -vertex tournament contains a copy of H .

For a **transitive tournament** \vec{T}_n ,

$$2^{n/2} \leq \vec{r}(\vec{T}_n) \leq 2^n.$$

The upper bound implies that $\vec{r}(H)$ exists for all **acyclic** H .

If H has εn^2 edges, then

$$\vec{r}(H) \geq 2^{\varepsilon n}.$$

Ramsey numbers of sparse undirected graphs

If H is a **tree** or **cycle**, then $r(H) = O(n)$.

Burr-Erdős (1975): Does $r(H) = O(n)$ for **all** sparse H ?

Theorem (Chvátal-Rödl-Szemerédi-Trotter 1983)

If H has n vertices and maximum degree Δ , then $r(H) = O_{\Delta}(n)$.

A more refined notion of sparsity is **degeneracy**, defined by

$$\max_{H' \subseteq H} (\text{minimum degree of } H').$$

If H is d -degenerate, then $r(H) \geq 2^{d/2}$. So graphs of unbounded degeneracy have “large” Ramsey numbers.

Conjecture (Burr-Erdős 1975), Theorem (Lee 2017)

If H is d -degenerate, then $r(H) = O_d(n)$.

Upshots: H has linear Ramsey number “if and only if” H is sparse.
Qualitatively, n and d control $r(H)$.

Ramsey numbers of sparse digraphs

Conjecture (Sumner 1971)

If H is any orientation of an n -vertex tree, then $\vec{r}(H) \leq 2n - 2$.

Häggkvist-Thomason (1991): $\vec{r}(H) \leq 12n$.

Kühn-Mycroft-Osthus (2011): $\vec{r}(H) \leq 2n - 2$ for $n \geq n_0$.

Theorem (Thomason 1986)

If H is any acyclic orientation of C_n , then $\vec{r}(H) = n$ for $n \geq n_0$.

Bucić-Letzter-Sudakov: Is $\vec{r}(H)$ linear for all **bounded-degree** H ?

Theorem (Yuster 2020, Girão 2020, DDFGHKLMSS 2020)

If H has bandwidth k , (i.e. there is an edge $v_i \rightarrow v_j$ only if $1 \leq j - i \leq k$) then $\vec{r}(H) = O_k(n)$.



Main results

Bucić-Letzter-Sudakov: Is $\vec{r}(H)$ linear for all **bounded-degree** H ?

No!

Theorem (Fox-He-W. 2021)

For all $C > 0$ and $n \geq n_0$, there is a **bounded-degree** ($\Delta \leq C^{3/2+o(1)}$) n -vertex acyclic digraph H with

$$\vec{r}(H) > n^C.$$

Theorem (Fox-He-W. 2021)

Let H be an n -vertex acyclic digraph with maximum degree Δ .

- $\vec{r}(H) \leq n^{O_\Delta(\log n)}$.
- If H has height h , then $\vec{r}(H) \leq n \cdot h^{O_\Delta(\log h)} = O_{\Delta,h}(n)$.
- If H is chosen randomly, then $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$ w.h.p.

Height (aka depth) = length of longest directed path

What determines if $\vec{r}(H)$ is large?

Recall: In the undirected setting, number of vertices and degeneracy determine how large $r(H)$ is.

What **additional** parameters are relevant in the directed setting?

If H is an acyclic digraph, we can order its vertices as v_1, \dots, v_n such that all edges go to the right ($v_i \rightarrow v_j$ implies $i < j$).

Given such an ordering, the *length* of an edge $v_i \rightarrow v_j$ is $j - i$.

“Definition”

If H has “many” edges of length in $[2^t, 2^{t+1})$ for “most” $0 \leq t \leq \log n$, then H has *high multiscale complexity*.

If not, H has *low multiscale complexity*.

“Theorem”

Let H be a bounded-degree acyclic digraph. Then $\vec{r}(H)$ is large “if and only if” H has **high multiscale complexity**.

Multiscale complexity affects $\vec{r}(H)$

Multiscale complexity: Many edges in many dyadic length scales.

“Theorem”

Let H be a bounded-degree acyclic digraph. Then $\vec{r}(H)$ is large “if and only if” H has high multiscale complexity.

- If H has bandwidth k , then every edge in H has length $\leq k$.
- If H has height h , then “most” edges have length in $[n/h, n]$.
- In a random digraph, a $o(1)$ fraction of edges have length $o(n)$.
- Our construction of a bounded-degree H with $\vec{r}(H) > n^c$ has many edges at every dyadic scale (“**interval mesh**”).

Conclusion and open questions

Let H have n vertices and maximum degree Δ .

- There is a gap between the n^C lower bound and $n^{O_\Delta(\log n)}$ upper bound on $\vec{r}(H)$.
We conjecture that the **upper bound** is closer to the truth.
- If H is **random**, we conjecture $\vec{r}(H) = O_\Delta(n)$ w.h.p., but can only prove $\vec{r}(H) \leq n(\log n)^{O_\Delta(1)}$.
This boils down to improving one technical lemma.
- Some notion of **multiscale complexity** affects whether $\vec{r}(H)$ is small or large.
 - ▶ Can one formalize this?
 - ▶ Which other digraph parameters are relevant?

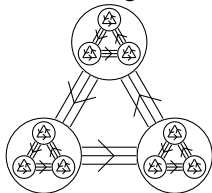
Lower bound proof sketch

Theorem

There exists an n -vertex acyclic digraph H with maximum degree ≤ 1000 and $\vec{r}(H) > n^{\log_2(3)-\epsilon}$.

We need (1) a construction of H , (2) a tournament T on $n^{\log_2(3)-\epsilon}$ vertices, and (3) a proof that there is no embedding $H \hookrightarrow T$.

For (2): We let T be an iterated blowup of a cyclic triangle.



For (3): Construct H so that in any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $\geq 0.49n$ is mapped into a single part.

Ensure that the induced subgraph on this subinterval has the same property, so we can iterate. At each step, $|T|$ drops by a factor of 3, but $|H|$ drops by a factor of 2.01.

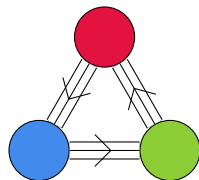
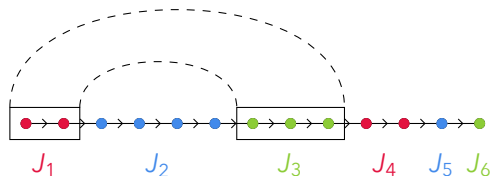
Lower bound proof sketch: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $\geq 0.49n$ is mapped into a single part, and this is **hereditary**.

Definition

H is an *interval mesh* if

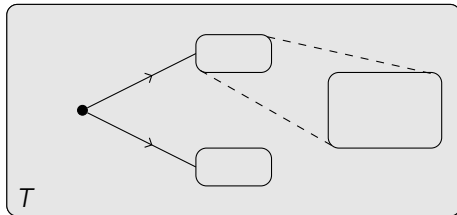
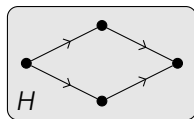
- H has a Hamiltonian path $1 \rightarrow 2 \rightarrow \dots \rightarrow n$.
- For all $1 \leq a < b \leq c < d \leq n$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge between $[a, b]$ and $[c, d]$.



Thus, $|J_i| > 100 \min(|J_{i-1}|, |J_{i+1}|)$. So $|J_i| \geq 0.49n$ for some i .

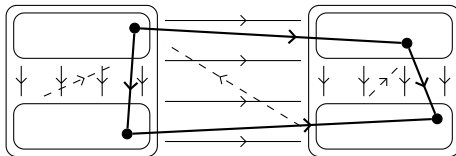
Greedy algorithm yields an interval mesh with max degree ≤ 1000 .

Upper bound proof sketch: greedy embedding



Lemma

If T is H -free, then T contains two large vertex sets with most edges between them oriented the same way.



The **multiscale complexity** of H controls the number of iterations.

Conclusion and open questions

Let H have n vertices and maximum degree Δ .

- There is a gap between the n^C lower bound and $n^{O_\Delta(\log n)}$ upper bound on $\vec{r}(H)$.
We conjecture that the **upper bound** is closer to the truth.
- If H is **random**, we conjecture $\vec{r}(H) = O_\Delta(n)$ w.h.p., but can only prove $\vec{r}(H) \leq n(\log n)^{O_\Delta(1)}$.
This boils down to improving one technical lemma.
- Some notion of **multiscale complexity** affects whether $\vec{r}(H)$ is small or large.
 - ▶ Can one formalize this?
 - ▶ Which other digraph parameters are relevant?

Thank you!