Ramsey numbers of sparse digraphs

Yuval Wigderson (Stanford)
Joint with Jacob Fox and Xiaoyu He

April 16, 2021

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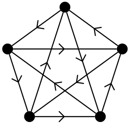

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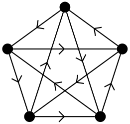

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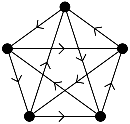

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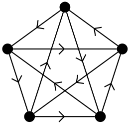

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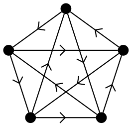

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Rédei's theorem $\iff \vec{r}(P_n) = n$, where $P_n =$ directed n-vertex path.

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So the Ramsey number is exponential if H is dense.

For the rest of the talk, we'll focus on sparse (di)graphs.

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Upshots: H has linear Ramsey number "if and only if" H is sparse.
Qualitatively, n and d control $r(H)$.

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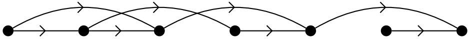

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- *If H is chosen randomly, then* $\vec{r}(H) \leq n \cdot (\log n)^{O_\Delta(1)}$ *w.h.p.*

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- In a random digraph, a $o(1)$ fraction of edges have length $o(n)$.
- Our construction of a bounded-degree H with $\vec{r}(H) > n^C$ has many edges at every dyadic scale ("interval mesh").

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  - Can one formalize this?

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- If H is random, we conjecture $\vec{r}(H) = O_\Delta(n)$ w.h.p., but can only prove $\vec{r}(H) \le n(\log n)^{O_\Delta(1)}$.
  This boils down to improving one technical lemma.
- Some notion of multiscale complexity affects whether $\vec{r}(H)$ is small or large.
  - Can one formalize this?
  - Which other digraph parameters are relevant?

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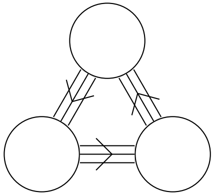

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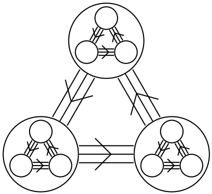

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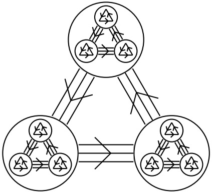

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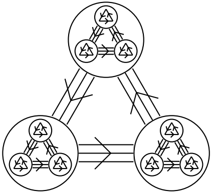

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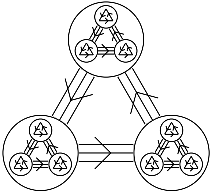

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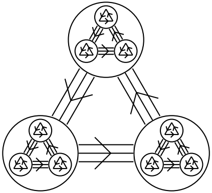

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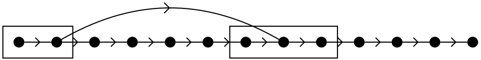

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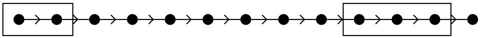

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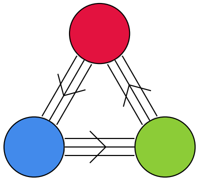

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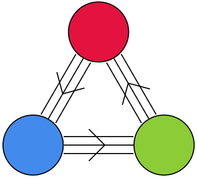

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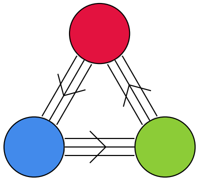

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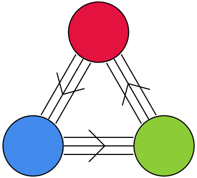

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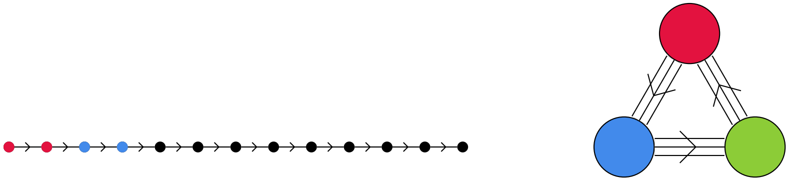

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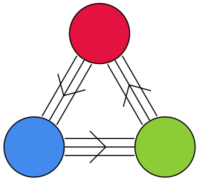

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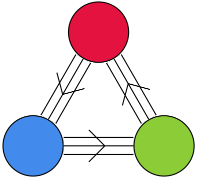

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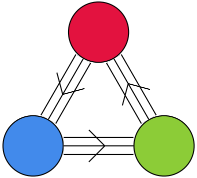

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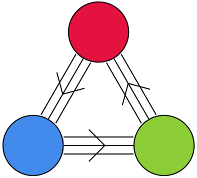

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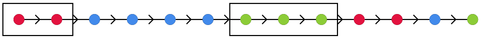

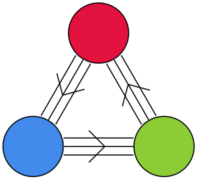

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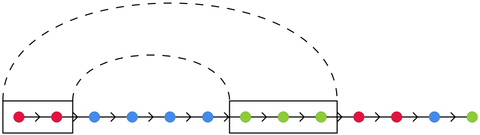

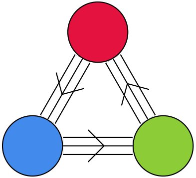

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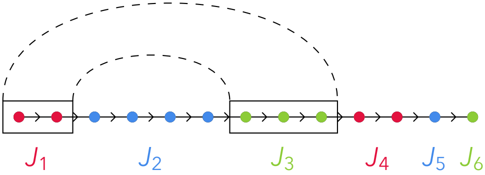

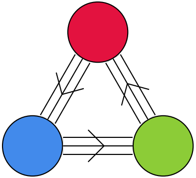

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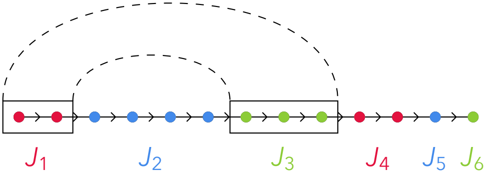

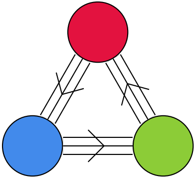

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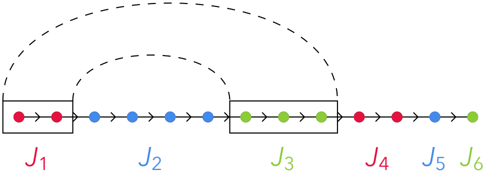

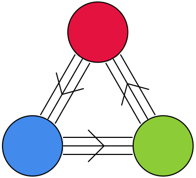

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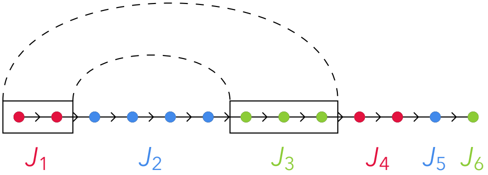

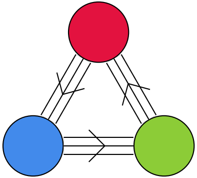

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Greedy algorithm yields an interval mesh with max degree ≤ 1000.

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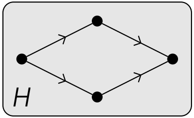

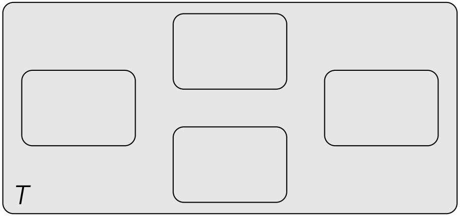

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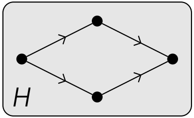

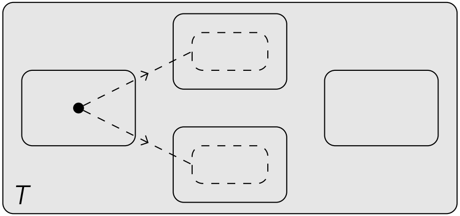

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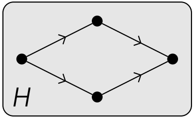

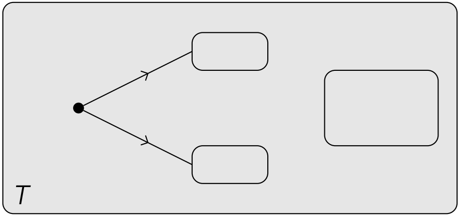

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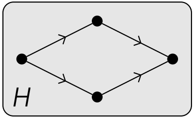

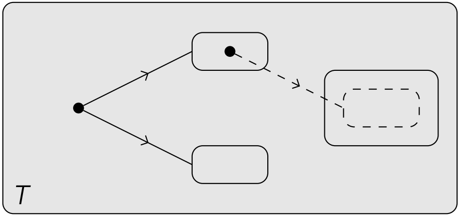

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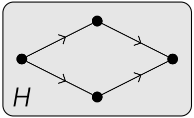
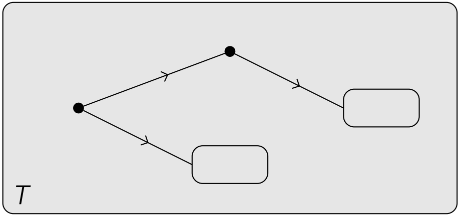
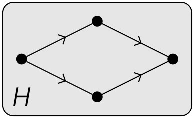
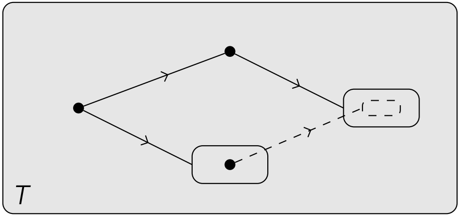




Upper bound proof sketch: greedy embedding





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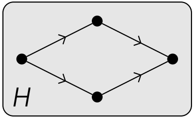

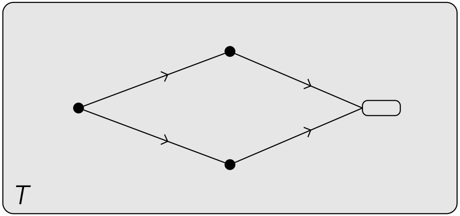

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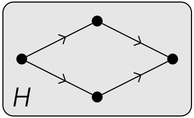

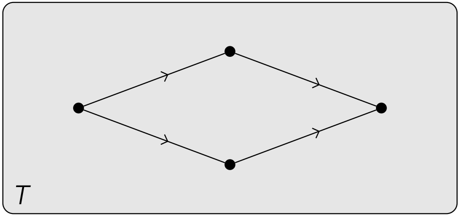

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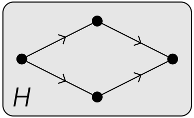

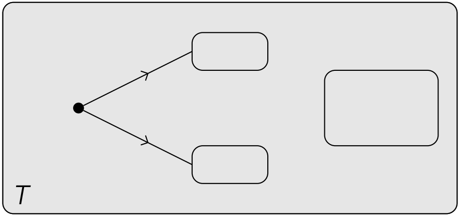

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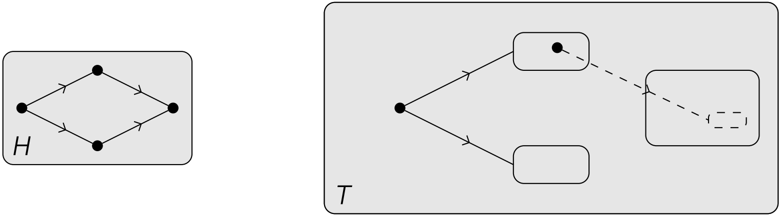

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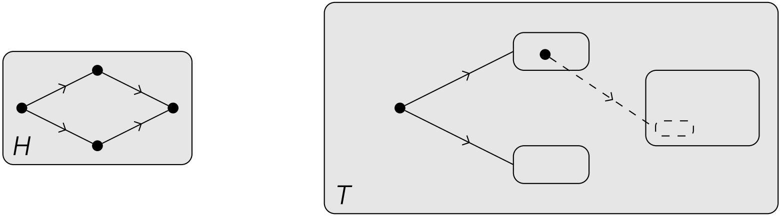

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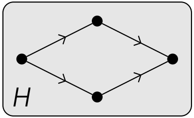

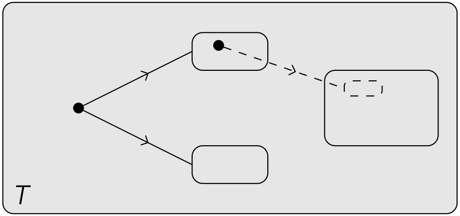

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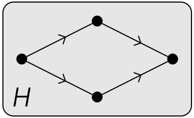

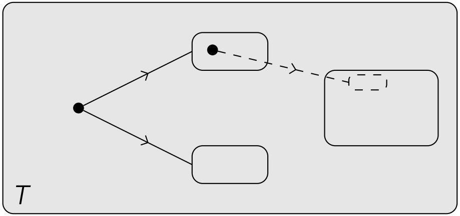

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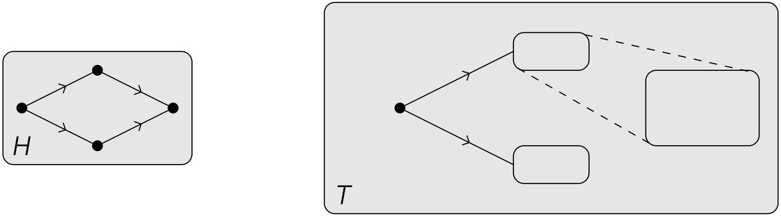

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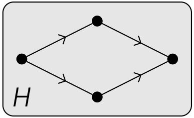
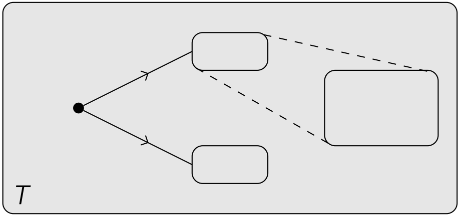
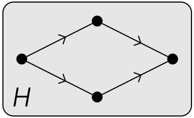


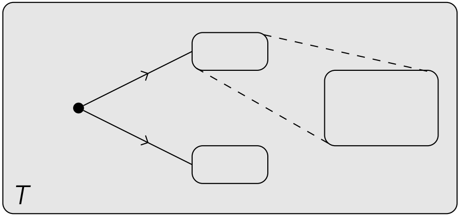


Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

Upper bound proof sketch: greedy embedding





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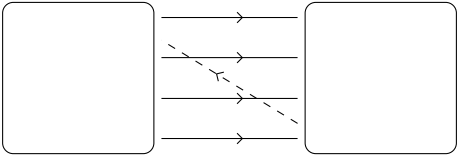

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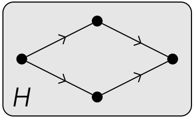

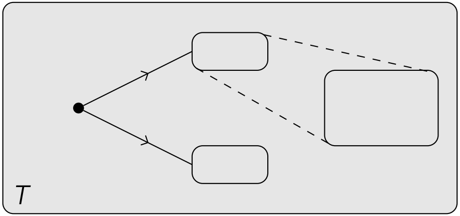

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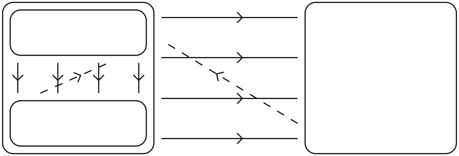

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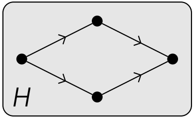

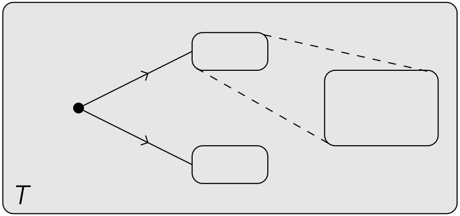

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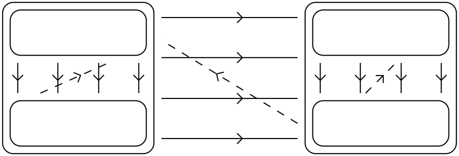

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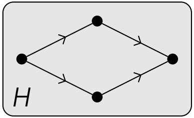

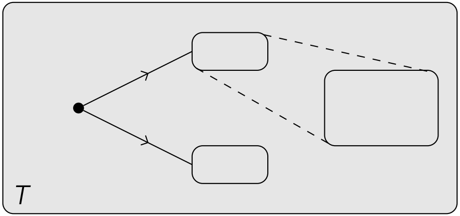

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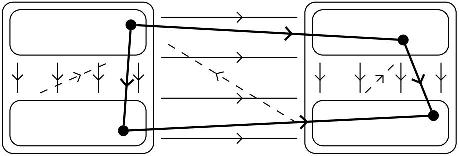

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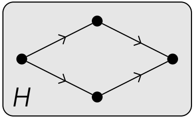

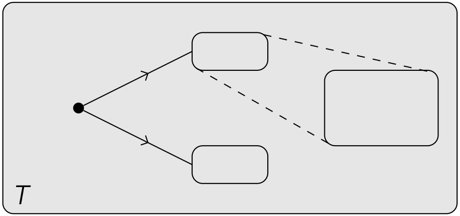

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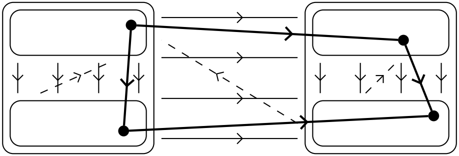

The multiscale complexity of *H* controls the number of iterations.

Conclusion and open questions

Let H have n vertices and maximum degree Δ.

- There is a gap between the n^C lower bound and $n^{O_\Delta(\log n)}$ upper bound on $\vec{r}(H)$.
  We conjecture that the upper bound is closer to the truth.
- If H is random, we conjecture $\vec{r}(H) = O_\Delta(n)$ w.h.p., but can only prove $\vec{r}(H) \leq n(\log n)^{O_\Delta(1)}$.
  This boils down to improving one technical lemma.
- Some notion of multiscale complexity affects whether $\vec{r}(H)$ is small or large.
  - Can one formalize this?
  - Which other digraph parameters are relevant?

Thank you!