"Normal" homework problems

✥ $\star$ 0. If you like historical fiction, you should consider reading the *Wolf Hall* trilogy or *A Place of Greater Safety* by Hilary Mantel. They have nothing to do with this class, except that her name is Mantel and the books are *really* good.

1. What does Turán's theorem mean in case $r = 2$? Is the theorem true in that case?

2. Find a general formula for $t_{r-1}(n)$, in terms of n, r, and $s \coloneqq n \pmod{r-1}$.

3. Prove that $T_{r-1}(n)$ maximizes number of edges among all complete $(r-1)$-partite graphs (that is, that any complete $(r-1)$-partite graph with parts of sizes *different* from $\lfloor n/(r-1)\rfloor$ or $\lceil n/(r-1)\rceil$ has fewer edges than $T_{r-1}(n)$).

4. Let G be an n-vertex graph. Recall that the *independence number* of G, denoted $\alpha(G)$, is the size of the largest set of vertices in G containing no edge. Let Δ be the maximum degree of G, and let d be the average degree of G.

   (a) Prove that $\chi(G) \le \Delta(G) + 1$. Conclude that $\alpha(G) \ge n/(\Delta + 1)$.

   (b) Using Turán's theorem, prove that $\alpha(G) \ge n/(d + 1)$. Note that this is a (much!) stronger result.

5. Let $\mathcal{H}$ be a collection of graphs. We say that G is $\mathcal{H}$-*free* if G has no copy of any $H \in \mathcal{H}$, and define
$$\mathrm{ex}(n, \mathcal{H}) = \max\{e(G) : G \text{ is an } n\text{-vertex } \mathcal{H}\text{-free graph}\}.$$
   Assuming the Erdős–Stone–Simonovits theorem, prove that
$$\mathrm{ex}(n, \mathcal{H}) = \left(1 - \frac{1}{\chi(\mathcal{H}) - 1} + o(1)\right)\binom{n}{2},$$
   where $\chi(\mathcal{H}) \coloneqq \min\{\chi(H) : H \in \mathcal{H}\}$.

$\star$ 6. Let G be an n-vertex triangle-free graph.

   (a) Suppose every vertex of G has degree greater than $2n/5$. Prove that G is bipartite.

   (b) Show that $2/5$ is the optimal constant in this theorem, that is, that for every n, there exists a non-bipartite triangle-free graph with minimum degree $\lfloor 2n/5\rfloor$.

   $\star\star$ (c) Can you find generalizations of parts (a) and (b) for K_r-free graphs, $r > 3$?

✥ $\star$ 7. Fix a probability distribution on $\mathbb{R}^d$, and let X, Y be two independent random vectors drawn according to this distribution. Prove that
$$\Pr(\|X + Y\| \ge 1) \ge \frac{1}{2}\Pr(\|X\| \ge 1)^2,$$
   where $\|\cdot\|$ denotes the usual Euclidean length of a vector.
   Hint: Mantel's theorem is useful here, though it doesn't look like it!

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$\star$ means that this problem is harder than the other ones.

Alternative proofs of Turán's theorem

8. Provide an alternative proof of Turán's theorem by induction on n (with inductive steps of size 1) by deleting a vertex of minimum degree.

9. Provide an alternative proof of Turán's theorem using a technique called *Zykov symmetrization*. Let G be a K_r-free n-vertex graph.

   (a) Pick two non-adjacent vertices $x, y \in V(G)$, and assume without loss of generality that $\deg(x) \geq \deg(y)$. Replace y with a *clone* of x, i.e. another vertex x' with the same neighborhood as x.

   (b) Repeat step (a) over and over until doing so no longer changes the graph (and prove that this must eventually happen).

   (c) Prove that the resulting graph when you get stuck is complete $(r-1)$-partite.

   (d) Conclude that $e(G) \leq t_{r-1}(n)$, with equality if and only if $G \cong T_{r-1}(n)$.

10. Provide an alternative proof of Turán's theorem using induction on r. Let G be a K_r-free n-vertex graph.

   (a) Let v be a vertex of maximum degree in G. Let A be the set of neighbors of v, and let $B = V(G) \setminus A$.

   (b) Form a new graph H by deleting all edges inside B, and adding in all missing edges between A and B. Prove that $e(H) \geq e(G)$.

   (c) Apply the inductive hypothesis (Turán's theorem for $r-1$) to the induced subgraph on A. Conclude that Turán's theorem holds for r.

⟴⋆ 11. Provide a ring-theoretic proof of Turán's theorem (you should skip this problem if you haven't seen ring theory).

   (a) Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. Define the *graph polynomial*

$$p_G(x_1, \ldots, x_n) = \prod_{\substack{i<j \\ v_i v_j \notin E(H)}} (x_i - x_j).$$

   Prove that G is K_r-free if and only if for all distinct $i_1, \ldots, i_r \in \{1, 2, \ldots, n\}$, we have that $p_G(x_1, \ldots, x_n) = 0$ if we set $x_{i_1} = x_{i_2} = \cdots x_{i_r}$.

   (b) Let J denote the set of all polynomials $p \in \mathbb{R}[x_1, \ldots, x_n]$ with the property that $p(x_1, \ldots, x_n) = 0$ whenever we set $x_{i_1} = x_{i_2} = \cdots x_{i_r}$, for any distinct $i_1, \ldots, i_r \in \{1, 2, \ldots, n\}$. Prove that J is an ideal.

   (c) Let I denote the ideal generated by all polynomials p_H, where H has chromatic number at most $r-1$. In other words,

$$I = \left\{ \sum_H \lambda_H p_H : \lambda_H \in \mathbb{R}, \text{ and } \chi(H) \leq r-1 \right\}.$$

   Prove that $I \subseteq J$.

   ⋆⋆ (d) Prove that $I = J$, using induction on n.

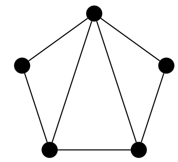
   ⋆ (e) Conclude Turán's theorem from part (d).

1. Today we proved that for any graph H,

$$\mathrm{ex}(n, H) \geq t_{\chi(H)-1}(n), \qquad (*)$$

which in particular implies the lower bound in the Erdős–Stone–Simonovits theorem. In this problem, you'll see examples of graphs where inequality $(*)$ is not best possible, i.e. where the Turán graph $T_{\chi(H)-1}(n)$ has strictly fewer edges than $\mathrm{ex}(n, H)$.

(a) Let H be the graph

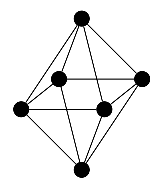


Verify that $\chi(H) = 3$, so that inequality $(*)$ implies $\mathrm{ex}(n, H) \geq t_2(n) = \lfloor n^2/4 \rfloor$.

(b) Add some edges to the Turán graph $T_2(n)$ to prove that

$$\mathrm{ex}(n, H) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor.$$

⋆ (c) Let O_3 be the graph corresponding to the octahedron, namely the graph



Verify that $\chi(O_3) = 3$. Using ideas discussed in class today, add edges to $T_2(n)$ to prove that

$$\mathrm{ex}(n, O_3) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + cn^{3/2}$$

for some absolute constant $c > 0$.

(d) Why don't these examples violate the Erdős–Stone–Simonovits theorem?

2. In this problem, you will study $\mathrm{ex}(n, T)$ in case T is a tree.

(a) Suppose that T is a tree with $t + 1$ vertices, and G is a graph with minimum degree at least t. Prove that G contains a copy of T.
Hint: Induction on t.

(b) Let G be an n-vertex graph with m edges. Prove that there is a subgraph $G' \subseteq G$ with minimum degree strictly greater than m/n.
Hint: Repeatedly delete vertices of degree $\leq m/n$.

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(c) Using parts (a) and (b), prove that if T is a tree with $t+1$ vertices, then
$$\mathrm{ex}(n, T) < (t-1)n.$$

(d) Prove that if n is divisible by t, then
$$\mathrm{ex}(n, T) \geq \frac{(t-1)n}{2}.$$

? (e) Erdős and Sós conjectured that the lower bound in part (d) is best possible, i.e. that
$$\mathrm{ex}(n, T) = \left\lceil \frac{(t-1)n}{2} \right\rceil$$
for all $(t+1)$-vertex trees T. Can you prove or disprove this conjecture?

3. Let $K_{1,r}$ denote the star with r leaves. Determine $\mathrm{ex}(n, K_{1,r})$ for all n and r. Is your answer consistent with the Erdős–Sós conjecture from the previous problem? Is it consistent with the Kővári–Sós–Turán theorem we proved in class?

⋆ 4. In this problem you'll prove the Erdős–Sós conjecture in the special case that T is a path. By the *length* of a path, we mean the number of vertices it has.

⋆⋆ (a) Let G be an n-vertex connected graph with minimum degree $\delta(G)$. Prove that G contains a path of length at least $\min\{n, 2\delta(G)+1\}$.
Hint: Consider a longest path in G, and try to extend it.

(b) Let P_{t+1} denote the path of length $t+1$. Prove that
$$\mathrm{ex}(n, P_{t+1}) \leq \left\lceil \frac{(t-1)n}{2} \right\rceil.$$
Hint: Induction on n.

⋆ (c) Can you characterize the extremal graphs, i.e. the P_{t+1}-free graphs with the maximum number of edges?

✣⋆ 5. Suppose $p_1, \ldots, p_n \in \mathbb{R}^2$ are n points in the plane. Prove that the number of *unit distances* among them (i.e. pairs $\{p_i, p_j\}$ with $\|p_i - p_j\| = 1$) is at most $O(n^{3/2})$.
Can you prove a stronger upper bound, or find a matching lower bound?

✣ 6. In this problem you will prove Jensen's inequality in full generality.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, we have that
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$
Prove that if f is twice-differentiable and satisfies $f'' \geq 0$, then f is convex.

(b) Suppose f is convex. Let $x_1, \ldots, x_n \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ with $\lambda_1 + \cdots + \lambda_n = 1$. Prove that
$$\sum_{i=1}^n \lambda_i f(x_i) \geq f\left(\sum_{i=1}^n \lambda_i x_i\right)$$
by induction on n. This is the general form of Jensen's inequality

(c) Prove that $f(x) = \binom{x}{r}$ is convex on the interval $[r, \infty)$ using part (a), and conclude the version of Jensen's inequality that I stated in class from part (b).

1. Recall that we defined
$$m_2(H) = \max_{F \subseteq H} \frac{e(F) - 1}{v(F) - 2},$$
and stated in class that $\mathrm{ex}(n, H) \geq \Omega(n^{2-1/m_2(H)})$ for all bipartite H.

   (a) Compute $m_2(C_{2\ell})$ for each $\ell \geq 2$. What lower bound on $\mathrm{ex}(n, C_{2\ell})$ do you get?

   (b) Compute $m_2(K_{s,t})$ for all $t \geq s \geq 2$. How does the resulting lower bound compare to the others we've discussed?

   (c) Compute $m_2(T)$ for any tree T. How does the resulting lower bound relate to the problems on yesterday's homework?

   $\star$ (d) Pick your favorite bipartite graph, and compute the lower and upper bounds coming from $m_2(H)$ and from finding H as a subgraph of $K_{s,t}$, respectively. Can you improve either of these bounds?

2. Using previous homework problems, prove the following fact. A graph H is a forest if and only if $\mathrm{ex}(n, H) = O(n)$.

$\star$ 3. In this problem, you'll prove that $\mathrm{ex}(n, H) \geq \Omega(n^{2-1/m_2(H)})$.

   (a) Let $p \in [0, 1]$, and let G be a *random* n-vertex graph obtained by making every pair of vertices adjacent with probability p, independently over all choices. Intuitively convince yourself (proving it is not super easy) that G should have roughly $p\binom{n}{2}$ edges with high probability.

   (b) Intuitively convince yourself (proving this is actually pretty hard) that for any fixed graph H, the number of copies of H in G is at most $O(n^{v(H)}p^{e(H)})$ with high probability.

   (c) Let $p = cn^{-1/m_2(H)}$, for some constant $c > 0$. Assuming parts (a) and (b), prove that if c is chosen appropriately, then $e(G) = \Omega(n^{2-1/m_2(H)})$ and that the number of copies of H in G is at most $\frac{1}{2}e(G)$, with high probability.

   (d) Delete one edge from each copy of H in G to get the desired result.

$\star$ 4. In this problem you'll examine extremal numbers of cycles.

   (a) Prove that if G is an n-vertex graph with $Cn^{1+1/\ell}$ edges (for some appropriate constant C), then G contains a cycle of length at most 2ℓ.

   $\star\star$ (b) Part (a) says that if $\mathcal{C}_{2\ell} = \{C_3, C_4, \ldots, C_{2\ell}\}$, then
$$\mathrm{ex}(n, \mathcal{C}_{2\ell}) \leq O(n^{1+1/\ell}).$$
   Strengthen this and prove that $\mathrm{ex}(n, C_{2\ell}) \leq O(n^{1+1/\ell})$.

$\star$ means that this problem is harder than the others. Also, stars are additive: two extra stars in a part of a starred multi-part problem correspond to three normal stars.

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(c) Let p be a prime, $2 \le \ell \le p$ a positive integer, and let $a_1, \dots, a_\ell$ be ℓ distinct elements of $\mathbb{F}_p$. Prove that the vectors

$$(1, a_1, a_1^2, \dots, a_1^{\ell-1}), \qquad (1, a_2, a_2^2, \dots, a_2^{\ell-1}), \qquad \dots \qquad (1, a_\ell, a_\ell^2, \dots, a_\ell^{\ell-1})$$

are linearly independent in $\mathbb{F}_p^\ell$.

$\star$ (d) Let p and ℓ be as above, and consider the following bipartite graph G. Its two parts are P and L, where $P = \mathbb{F}_p^\ell$ and L consists of all lines in $\mathbb{F}_p^\ell$ of the form

$$\{(b_1, \dots, b_\ell) + t \cdot (1, a, a^2, \dots, a^{\ell-1}) : t \in \mathbb{F}_p\}.$$

Make $p \in P$ and $\ell \in L$ adjacent in G if and only if $p \in \ell$. Prove that G has $n = 2p^\ell$ vertices and $p^{\ell+1} = \Theta(n^{1+1/\ell})$ edges. Moreover, prove that if $p \in \{2, 3, 5\}$, then G is $C_{2\ell}$-free. What goes wrong if $\ell \notin \{2, 3, 5\}$?

5. Recall that we stated a hypergraph version of the Kővári–Sós–Turán theorem, and proved it (at least in the case $k = 3$) by induction on k. Try proving the $k = 2$ case (i.e. the original Kővári–Sós–Turán theorem) via a similar inductive approach. What does the $k = 1$ case even mean?

$\star\star$ 6. In this problem, you will prove the following amazing strengthening of the Kővári–Sós–Turán theorem: if H is a bipartite graph and every vertex on one side has degree at most s, then $\mathrm{ex}(n, H) = O(n^{2-1/s})$.

$\star\star$ (a) Prove the following lemma. For all positive integers a, b, there exists some constant $C > 0$ such that the following holds. Let G be an n-vertex graph with average degree $d \ge Cn^{1-1/s}$. Then there exists $U \subseteq V(G)$ with $|U| \ge a$ so that every s-tuple of vertices in U has at least $a + b$ common neighbors.

Hint: Pick $x_1, \dots, x_s$ to be uniformly random vertices of G, chosen with repetition, and let X be the common neighborhood of $x_1, \dots, x_s$. The desired U can be obtained by deleting some vertices from X, with positive probability.

(b) Using the lemma, prove that $\mathrm{ex}(n, H) = O(n^{2-1/s})$ if every vertex on one side of H has degree at most s.

$\star$ 7. Recall that $K_r^{(k)}$ denotes the complete k-uniform hypergraph with r vertices.

(a) Prove that $\mathrm{ex}(n, K_4^{(3)}) \ge (\frac{5}{9} + o(1))\binom{n}{3}$.
Hint: Split the vertex set into three equal-sized parts.

$\star$ (b) Prove that $\mathrm{ex}(n, K_r^{(3)}) \ge (1 - \left(\frac{2}{r-1}\right)^2 + o(1))\binom{n}{3}$.

$\star$ (c) Prove that

$$\mathrm{ex}(n, K_r^{(k)}) \le \left(1 - \frac{1}{\binom{r}{k}} + o(1)\right)\binom{n}{k}.$$

$\star\star$ (d) Prove the best known upper bound on $\mathrm{ex}(n, K_r^{(k)})$, namely

$$\mathrm{ex}(n, K_r^{(k)}) \le \left(1 - \frac{1}{\binom{r-1}{k-1}} + o(1)\right)\binom{n}{k}.$$

? (e) Improve any of the bounds above.

1. Let $\mathcal{F}$ be a finite collection of bipartite graphs. A famous conjecture of Erdős and Simonovits, called the *compactness conjecture*, asserts that there is an absolute constant $C > 0$ (depending only on $\mathcal{F}$) so that

$$\mathrm{ex}(n, \mathcal{F}) \le \min_{H \in \mathcal{F}} \mathrm{ex}(n, H) \le C \cdot \mathrm{ex}(n, \mathcal{F}).$$

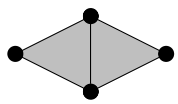
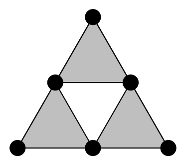
   (a) Prove the first inequality above.

   $\star$ (b) Prove that the second inequality can be false if we allow $\mathcal{F}$ to be infinite.

   ? (c) Prove or disprove the compactness conjecture.

   $\star\star$ (d) The compactness conjecture is known to be false for hypergraphs! You'll see this in this part and the next.

   Consider the following two 3-partite 3-graphs:

   $K^{(3)}_{1,1,2} =$  $T =$ 

   Prove that $\mathrm{ex}(n, K^{(3)}_{1,1,2}) = \Theta(n^2)$ and $\mathrm{ex}(n, T) = \Theta(n^2)$.

   $\star\star\star\star\star$ (e) Prove that $\mathrm{ex}(n, \{K^{(3)}_{1,1,2}, T\}) = o(n^2)$, thus disproving the compactness conjecture for hypergraphs.

   Remark: This is probably impossible for you to do with the techniques you know. But come to my Szemerédi class next week to find out how to do it!

2. For a k-graph $\mathcal{H}$ and an integer n, let $\pi_n(\mathcal{H}) := \mathrm{ex}(n, \mathcal{H})/\binom{n}{k}$.

   $\star$ (a) Prove that $\pi_n(\mathcal{H}) \ge \pi_{n+1}(\mathcal{H})$ for all n. Conclude that $\pi(\mathcal{H}) := \lim_{n\to\infty} \pi_n(\mathcal{H})$ is well-defined. $\pi(\mathcal{H})$ is called the *Turán density* of $\mathcal{H}$.

   (b) Let H be a graph (i.e. $k = 2$). What is $\pi(H)$?

3. For a graph H and an integer s, we denote by $H[s]$ the *s-blowup* of H. This is the graph obtained by replacing every vertex of H by an independent set of size s, and replacing every edge of s by a copy of $K_{s,s}$. Similarly, if $\mathcal{H}$ is a k-graph, then $\mathcal{H}[s]$ is the k-graph obtained by replacing every vertex by s vertices, and replacing every edge by a copy of $K^{(k)}_{s,\ldots,s}$.

   (a) Check that if $H = K_k$, our two definitions of $K_k[s]$ coincide.

   $\star$ (b) Prove the following general form of the supersaturation theorem.

   For every k-graph $\mathcal{H}$ and every $\varepsilon > 0$, there exists some $\delta > 0$ so that the following holds for all sufficiently large n. If $\mathcal{G}$ is an n-vertex k-graph with

$$e(\mathcal{G}) \ge (\pi(\mathcal{H}) + \varepsilon)\binom{n}{k}$$

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then $\mathcal{G}$ has at least $\delta\binom{n}{v(\mathcal{H})}$ copies of $\mathcal{H}$.

(c) Deduce from this the following general form of the Erdős–Stone theorem.

For every k-graph $\mathcal{H}$ and every positive integer s, we have $\pi(\mathcal{H}[s]) = \pi(\mathcal{H})$.

⊕ 4. In this problem, you will see another example of the supersaturation phenomenon, this time in the field of Ramsey theory.

⋆ (a) Ramsey's theorem says the following. For every positive integer k, there exists some positive integer R so that the following holds. No matter how we color the edges of K_R using the colors red and blue, there is a copy of K_k all of whose edges have the same color.

Prove Ramsey's theorem (or take it as a given and move onto the next part).

(b) Prove the following supersaturation version of Ramsey's theorem, which is usually called a *Ramsey multiplicity* result.

For every positive integer k, there exists some $\delta > 0$ so that the following holds for every sufficiently large n. No matter how we color the edges of K_n using the colors red and blue, there are at least $\delta\binom{n}{k}$ copies of K_k all of whose edges have the same color.

⋆⋆ (c) What sorts of bounds can you prove on δ in terms of k?

5. In this problem, you will prove a supersaturation result for complete bipartite graphs.

(a) Given two graphs H, G, a *graph homomorphism* from H to G is a function $f : V(H) \to V(G)$ with the property that if uv is an edge of H, then $f(u)f(v)$ is an edge of G. Note that if f is injective, then this yields a copy of H in G. If f is not injective, we say this is a *pseudocopy*.

Prove that if $v(G) = n$, then there are at most $n^{v(H)}$ homomorphisms from H to G, and at most $\binom{v(H)}{2}n^{v(H)-1}$ pseudocopies of H in G.

⋆ (b) Suppose G has n vertices and $pn^2/2$ edges (we say that G has *edge density* p). Prove that there are at least $p^{st}n^{s+t}$ homomorphisms from $K_{s,t}$ to G.

Hint: Use Jensen's inequality twice.

(c) Deduce from parts (a) and (b) the following supersaturation result. For every $\varepsilon > 0$ and integers s, t, there exists a $\delta > 0$ so that the following holds for sufficiently large n. If G has n vertices and $\varepsilon\binom{n}{2}$ edges, then G has at least $\delta\binom{n}{s+t}$ copies of $K_{s,t}$.

⋆ (d) Can you prove analogous results for k-uniform hypergraphs?

⋆⋆ 6. In class we proved the following sampling lemma: If G is an n-vertex graph with $e(G) \geq \beta\binom{n}{2}$, then the number of m-sets of vertices M with $e(M) \geq \alpha\binom{m}{2}$ is at least $(\beta - \alpha)\binom{n}{m}$. In fact, the proof showed that we could replace $\beta - \alpha$ above with $(\beta - \alpha)/(1 - \alpha)$.

Is this result best possible, or close to best possible? That is, is it true that for arbitrarily large n, there exists some n-vertex graph with $e(G) \approx \beta\binom{n}{2}$ and roughly $\frac{\beta-\alpha}{1-\alpha}\binom{n}{m}$ m-sets M with $e(M) \geq \alpha\binom{n}{m}$?

For concreteness, feel free to fix your favorite values of α, β, e.g. $\alpha = 1/3$ and $\beta = 2/3$. So can you find a sequence of graphs with around $\frac{2}{3}\binom{n}{2}$ edges so that roughly $\frac{1}{2}\binom{n}{m}$ of the m-sets M satisfy $e(M) \geq \frac{1}{3}\binom{m}{2}$?