Ancient Greek Calculus

Mathcamp 2020

Outline

Precalculus

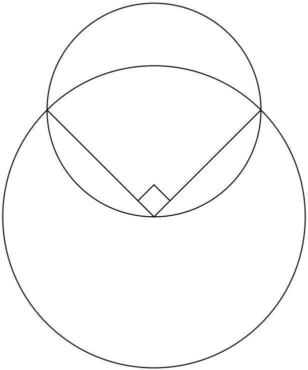
AP Calculus AB: Eudoxus

AP Calculus BC: Archimedes

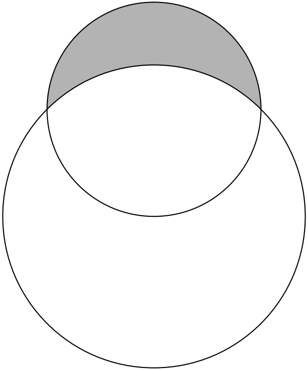
Outline

Hippocrates of Chios and his Lune

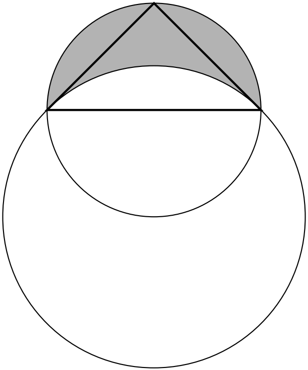
Hippocrates of Chios and his Lune



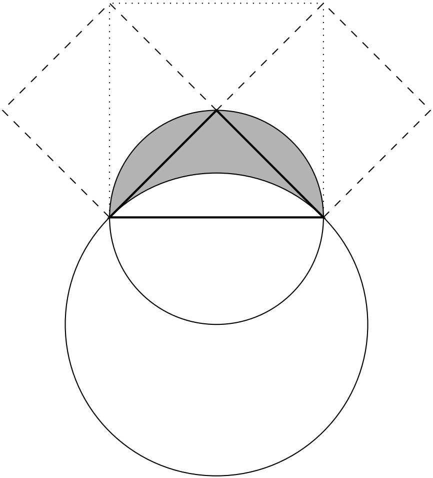
Hippocrates of Chios and his Lune



Hippocrates of Chios and his Lune

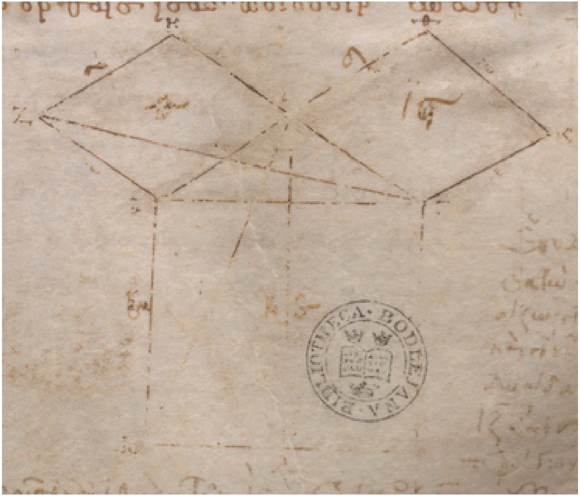


Hippocrates of Chios and his Lune

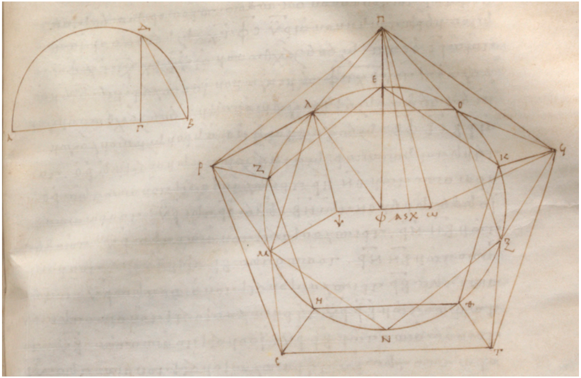


Greek geometric figures

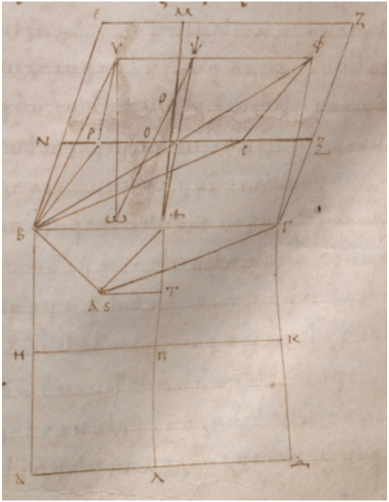
Greek geometric figures



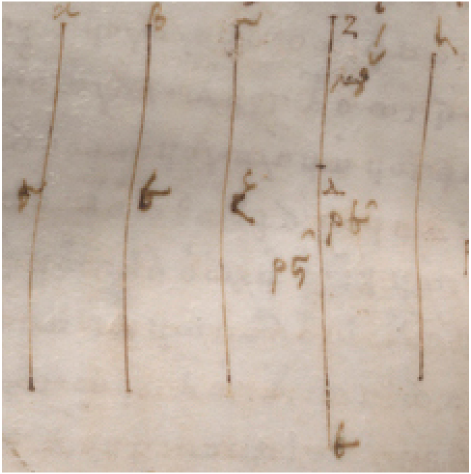
Greek geometric figures



Greek geometric figures



Greek number-theoretic figures



Greek number-theoretic figures

Basic facts from proportion theory

Basic facts from proportion theory

- If $A : B :: C : D$, then $A : C :: B : D$

Basic facts from proportion theory

- If $A : B :: C : D$, then $A : C :: B : D$
  i.e. if $a/b = c/d$, then $a/c = b/d$

Basic facts from proportion theory

- If $A : B :: C : D$, then $A : C :: B : D$
  i.e. if $a/b = c/d$, then $a/c = b/d$
- If $A : B :: C : D$ and A, B, C, D are lengths, then the rectangle with side lengths A, C has the same area as the rectangle with side lengths B, D

Basic facts from proportion theory

- If $A : B :: C : D$, then $A : C :: B : D$
  i.e. if $a/b = c/d$, then $a/c = b/d$
- If $A : B :: C : D$ and A, B, C, D are lengths, then the rectangle with side lengths A, C has the same area as the rectangle with side lengths B, D
  i.e. if $a/b = c/d$, then $ac = bd$

Basic facts from proportion theory

- If $A : B :: C : D$, then $A : C :: B : D$
  i.e. if $a/b = c/d$, then $a/c = b/d$
- If $A : B :: C : D$ and A, B, C, D are lengths, then the rectangle with side lengths A, C has the same area as the rectangle with side lengths B, D
  i.e. if $a/b = c/d$, then $ac = bd$

Goal: Prove such results for arbitrary ratios

Outline

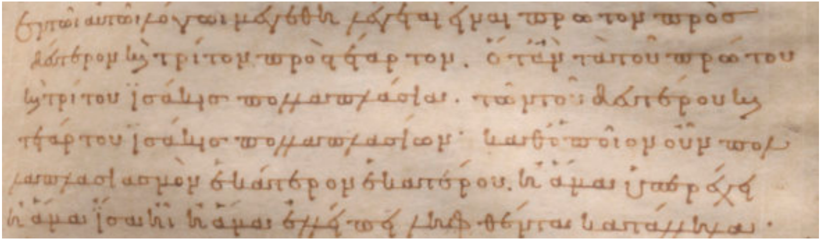
Precalculus

AP Calculus AB: Eudoxus

AP Calculus BC: Archimedes

A theory of the real numbers

A theory of the real numbers



A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d?

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d? For any integers m, n, consider ma, mc, nb, nd.

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d? For any integers m, n, consider ma, mc, nb, nd.

$$ma < nb \iff mc < nd$$

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d? For any integers m, n, consider ma, mc, nb, nd.

$$ma = nb \iff mc = nd$$

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d? For any integers m, n, consider ma, mc, nb, nd.

$$ma > nb \iff mc > nd$$

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d? For any integers m, n, consider ma, mc, nb, nd.

$$ma \begin{smallmatrix}\le\\=\\>\end{smallmatrix} nb \iff mc \begin{smallmatrix}\le\\=\\>\end{smallmatrix} nd$$

A theory of the real numbers

Definition (*Elements* V.5)

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." (trans. Thomas Little Heath)

When is a/b equal to c/d? For any integers m, n, consider ma, mc, nb, nd.

$$ma \mathrel{\substack{<\\=\\>}} nb \iff mc \mathrel{\substack{<\\=\\>}} nd$$

i.e. $a/b = c/d$ iff for all m, n,

$$\frac{a}{b} \mathrel{\substack{<\\=\\>}} \frac{n}{m} \iff \frac{c}{d} \mathrel{\substack{<\\=\\>}} \frac{n}{m}.$$

The method of exhaustion

The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$.

The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

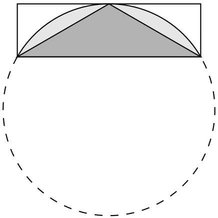
The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.



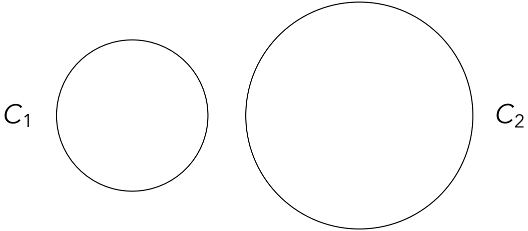
The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.



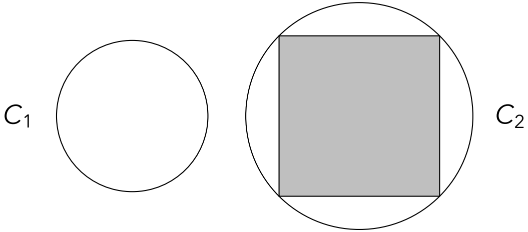
The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.



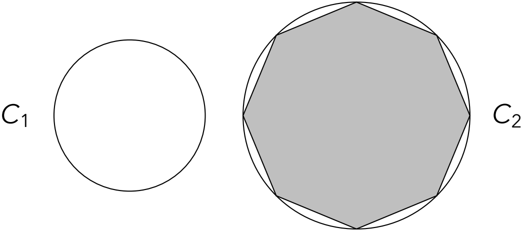
The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least 1/2 the area of the circular segment.



The method of exhaustion

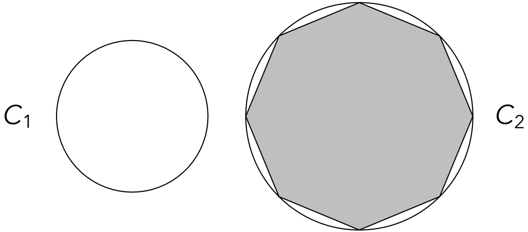
Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

remaining area
$< a_2 - x$



The method of exhaustion

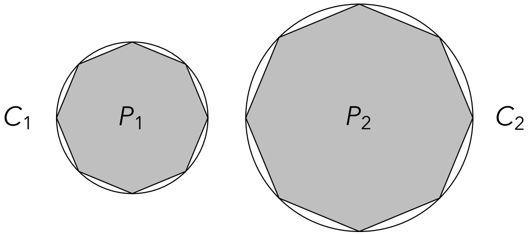
Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

$a_2 - p_2 < a_2 - x$



The method of exhaustion

Theorem (*Elements* XII.2)

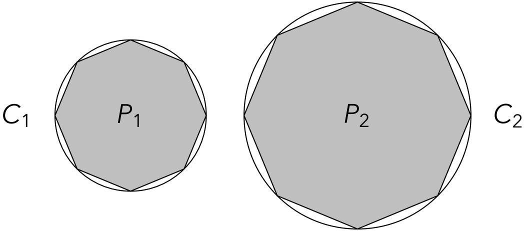
If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$.
Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

$a_2 - p_2 < a_2 - x$

$p_2 > x$



The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

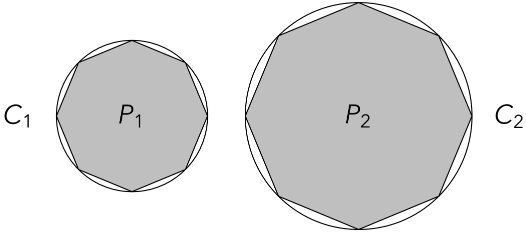
Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

$$a_2 - p_2 < a_2 - x$$

$$p_2 > x$$

$$\frac{p_1}{p_2} = \left(\frac{d_1}{d_2}\right)^2 = \frac{a_1}{x}$$



The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$.
Suppose first that $x < a_2$.

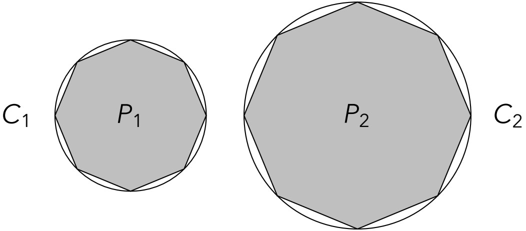
Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

$a_2 - p_2 < a_2 - x$

$p_2 > x$

$$\frac{p_1}{p_2} = \left(\frac{d_1}{d_2}\right)^2 = \frac{a_1}{x}$$

$$\frac{x}{p_2} = \frac{a_1}{p_1}$$



The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

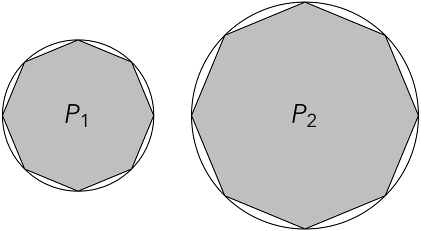
$a_2 - p_2 < a_2 - x$

$p_2 > x$

$$\frac{p_1}{p_2} = \left(\frac{d_1}{d_2}\right)^2 = \frac{a_1}{x}$$

$$\frac{x}{p_2} = \frac{a_1}{p_1}$$

C_1



C_2

But $a_1 > p_1$, so $x > p_2$!

The method of exhaustion

Theorem (*Elements* XII.2)

If two circles C_1, C_2 have diameters d_1, d_2 and areas a_1, a_2, then $a_1/a_2 = (d_1/d_2)^2$.

Proof by contradiction. Fix $x \neq a_2$ such that $a_1/x = (d_1/d_2)^2$. Suppose first that $x < a_2$.

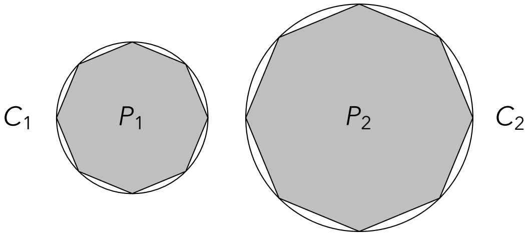
Lemma: If we bisect an arc of a circle, the area of the triangle is at least $1/2$ the area of the circular segment.

$a_2 - p_2 < a_2 - x$

$p_2 > x$

$$\frac{p_1}{p_2} = \left(\frac{d_1}{d_2}\right)^2 = \frac{a_1}{x}$$

$$\frac{x}{p_2} = \frac{a_1}{p_1}$$



But $a_1 > p_1$, so $x > p_2$! Swap C_1 and C_2 to deal with $x > a_2$.

Outline

Precalculus

AP Calculus AB: Eudoxus

AP Calculus BC: Archimedes

Archimedes

Archimedes

- Area vs. circumference of a circle

Archimedes

- Area vs. circumference of a circle
- $$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

Archimedes

- Area vs. circumference of a circle
- $$3\frac{10}{71} < \pi < 3\frac{1}{7}$$
- Area of a parabolic segment

Archimedes

- Area vs. circumference of a circle
- $$3\frac{10}{71} < \pi < 3\frac{1}{7}$$
- Area of a parabolic segment
- Center of mass of a parabolic segment

Archimedes

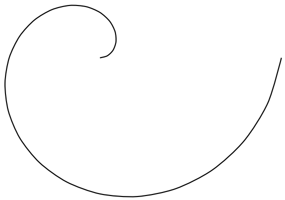
- Area vs. circumference of a circle
- $$3\frac{10}{71} < \pi < 3\frac{1}{7}$$
- Area of a parabolic segment
- Center of mass of a parabolic segment
- Volume and surface area of cylinder vs. sphere

Archimedes

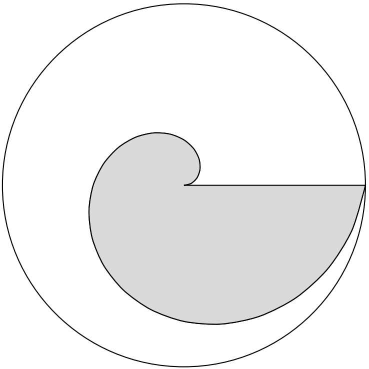
- Area vs. circumference of a circle
- $$3\frac{10}{71} < \pi < 3\frac{1}{7}$$
- Area of a parabolic segment
- Center of mass of a parabolic segment
- Volume and surface area of cylinder vs. sphere
- {Volumes, centers of mass} of {ellipsoids, paraboloids, hyperboloids} of revolution

Archimedes's spiral

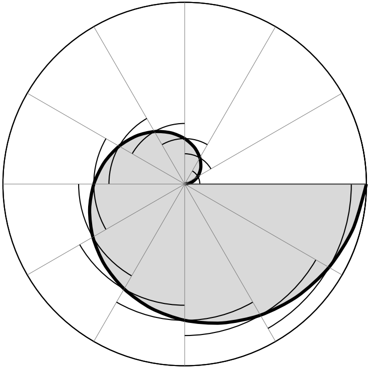
Archimedes's spiral



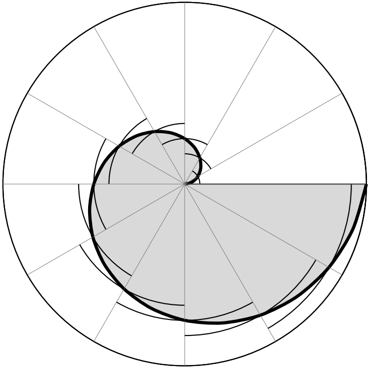
Archimedes's spiral



Archimedes's spiral



Archimedes's spiral



Since all the sectors have the same angle, their areas are proporitional to their radius squared. So this all boils down to

$$\sum_{i=1}^{n} i^2 \quad \rightsquigarrow \quad \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}$$