Ham Sandwich Theorem Problems #1

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Mathcamp 2015

1. In some textbooks, you'll see the following statement referred to as the Borsuk-Ulam Theorem:

   Theorem (Borsuk-Ulam). *Let $f : S^n \to \mathbb{R}^n$ be a continuous* ***odd*** *map. Then there is some $x \in S^n$ for which $f(x) = 0$.*

   Prove that this statement is equivalent to the Borsuk-Ulam Theorem presented in class.

2. Prove that n sets is indeed the most we can cut with a hyperplane in $\mathbb{R}^n$. In other words, find $n+1$ sets in $\mathbb{R}^n$ that cannot be simultaneously bisected by a hyperplane, and prove that they cannot.

3. Prove that there are exactly $\binom{n+d}{n}$ monomials with coefficient 1 of degree $\leq d$ on n variables.

4*. In this problem, we will prove the Borsuk-Ulam Theorem.

   (a) We say that two continuous maps $g, h : S^n \to S^n$ are *homotopic* if we can continuously deform g to get to h. More precisely, we say that g and h are homotopic if there is a continuous function $H : S^n \times [0,1] \to S^n$ with the property that $H(x,0) = g(x)$ and $H(x,1) = h(x)$ for all $x \in S^n$. In other words, we can think of the second variable of H as a time parameter, and when we are at time 0, we have the function g, while we have the function h at time 1.

   Prove that if both g and h are constant maps (i.e. each one has a one-point image), then g and h are homotopic. With this in mind, we say that a map is *nullhomotopic* if it is homotopic to any constant map.

   (b) An important fact from algebraic topology is that every continuous map $g : S^n \to S^n$ has a *degree*, which is an integer denoted by $\deg g$. The degree has the following properties:

   - $\deg(h \circ g) = (\deg h) \cdot (\deg g)$.
   - Two maps are homotopic if and only if they have the same degree[1].

   Example. As an example of how degrees actually work, consider S^1 to lie in the complex plane and to consist of all $z \in \mathbb{C}$ for which $|z| = 1$. Then the kth-power map that sends $z \mapsto z^k$ has degree k. Note that this map acts on the circle by simply "wraps around k times," which is why it's sometimes helpful to think of the degree of g as the number of times g "wraps around" S^n.

   Prove that any nullhomotopic map has degree 0 and that the identity map has degree 1. Note that this agrees with our intuition about "wrapping around" S^n.

   (c) Consider S^{n-1} as the equator of S^n, and let S^n_+ denote the northern hemisphere of S^n. Prove that any continuous map $g : S^{n-1} \to S^{n-1}$ is nullhomotopic if and only if it can be extended continuously to a map $\widetilde{g} : S^n_+ \to S^{n-1}$.

   (d) The final tool that we need is often called Borsuk's Lemma:

   Lemma (Borsuk). *Let $f : S^n \to S^n$ be an odd continuous map. Then f has odd degree.*

[1] It's important to realize that this is surprising. This tells us that if we only care about maps up to homotopy, then there are precisely $\mathbb{Z}$-many such maps.

Check that this lemma is correct for the kth-power map defined above. Using your intuition about degree measuring the number of times a map wraps around S^n, convince yourself that this lemma is reasonable.

(e) Suppose we had a continuous odd map $f : S^n \to \mathbb{R}^n$, and suppose that f never attains the value 0. Because of this, $|f(x)|$ is never 0 either, so we can define $g(x) = f(x)/|f(x)|$ without breaking Rule 4. Prove that g is a continuous map $S^n \to S^{n-1}$.

(f) Let g_0 denote the restriction of g to the equator S^{n-1} of S^n. Prove that g_0 is an odd continuous map $S^{n-1} \to S^{n-1}$. Also note that g_0 extends continuously to S^n_+, since g extends it.

(g) Derive a contradiction by applying both part (4c) and Borsuk's Lemma, along with the fact that a nullhomotopic map has degree 0. Conclude that the statement of the Borsuk-Ulam Theorem presented in Problem 1 is true, and thus prove the Borsuk-Ulam Theorem.

Ham Sandwich Theorem Problems #2

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1. Prove that the bound in the Polynomial Ham Sandwich Theorem is also tight, in the sense that we cannot bisect more than $\binom{n+d}{d}-1$ sets in $\mathbb{R}^n$ with a polynomial of degree $\leq d$.

2*. Prove rigorously that the function f defined in the proof of the Polynomial Ham Sandwich Theorem really is continuous.

3*. In this problem, we investigate a bit more the idea that we can cut as many sets as degrees of freedom we have available. Here is a more general theorem than the Polynomial Ham Sandwich Theorem:

Theorem. *Let $m \geq 1$ be some integer. Let $g_0, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ be $m+1$ arbitrary continuous functions, and suppose that $g_0, \dots, g_m$ are linearly independent (meaning that $\lambda_0 g_0 + \cdots + \lambda_m g_m$ is the constant 0 function if and only if $\lambda_0 = \cdots = \lambda_m = 0$), and let $F_1, \dots, F_m \subset \mathbb{R}^n$ be m bounded sets. Then there exist $a_0, \dots, a_m \in \mathbb{R}$ for which the set $\{a_0 g_0 + \cdots + a_m g_m = 0\}$ bisects each F_i simultaneously.*

Observe that this theorem generalizes the Polynomial Ham Sandwich Theorem by setting $m = \binom{n+d}{n}-1$ and $g_0, \dots, g_m$ the m monomial functions. Also observe that this gives us really cool results, such as that we can cut $n+1$ sets in $\mathbb{R}^n$ if we allow ourselves spherical knives (by setting $g_0 = 1$, $g_i = x_i$ for $1 \leq i \leq n$, and $g_{n+1} = x_1^2 + \cdots + x_n^2$).

Prove this theorem. In your proof, where do you use linear independence? Note that we really do have m degrees of freedom, and can cut precisely m sets.

4. Prove that in the Spice Sandwich Theorem, we in general can't avoid having some of our spice points lie on the bisecting hyperplane.

5. In proving the Spice Sandwich Theorem, we needed to know that the function m_* is continuous. Rigorously prove that m_* is continuous, using the $\varepsilon - \delta$ definition of continuity.

6. Prove that the Veronese mapping Φ is injective. Because of this, you will sometimes see it called the *Veronese embedding*.

7. Morally, we should be able to prove the Spice Sandwich Theorem in the following way: we place a small ball around each of our spice points, bisect this set (in the volume sense) using the Ham Sandwich Theorem, and finally shrink the radius of the balls to 0 in order to conclude that we have actually bisected the points.

   (a) What is the problem with this argument?

   (b)** Can you eliminate the problem and make this argument work?

8. (a) Try to prove the Polynomial Ham Sandwich Theorem using the Veronese mapping trick, like we did for the Polynomial Spice Theorem. Where does a difficulty arise?

   (b)** Can you resolve this difficulty and prove the Polynomial Ham Sandwich Theorem?

Ham Sandwich Theorem Problems #3

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Problems from Yesterday

These are problems from yesterday's problem set whose material we didn't cover until today.

1. Prove that the Veronese mapping Φ is injective. Because of this, you will sometimes see it called the *Veronese embedding*.

2. (a) Try to prove the Polynomial Ham Sandwich Theorem using the Veronese mapping trick, like we did for the Polynomial Spice Theorem. Where does a difficulty arise?

   (b)** Can you resolve this difficulty and prove the Polynomial Ham Sandwich Theorem?

Today's Problems

1*. State and prove the partitioning polynomial theorem in n dimensions. What bound do you get for $\deg Q$? (*Hint:* It's no longer just $O(\sqrt{r})$.)

2. Prove, without using Szemerédi-Trotter, that

$$I(C,L) \le O(mn^{1/2}+n) \qquad \text{and} \qquad I(C,L) \le O(m^{1/2}n+m)$$

Feel free to only prove one of these two bounds, since their proofs are very similar.

Hint: Let $\chi : C \times L \to \{0,1\}$ denote the indicator function defined by

$$\chi(c,\ell) = \begin{cases} 1 & \ell \text{ is incident to } c \\ 0 & \text{otherwise} \end{cases}$$

Then write

$$I(C,L)^2 = \left(\sum_{\ell\in L}\sum_{c\in C}\chi(c,\ell)\right)^2$$

and apply the Cauchy-Schwarz inequality.

3. Observe that the proposition we proved in class (that $I(C,L) \le m+n^2$) and the bounds in Exercise 2 don't really use many properties of points and lines—in fact, they rely only on the property that any two points define a unique line. This property is true in many geometric settings that are not the plane.

   (a) Recall that a *field* is an algebraic structure where we can add, subtract, multiply, and divide[1]. Well-known examples of fields are $\mathbb{R}, \mathbb{C}$, and $\mathbb{Q}$. Given any field $\mathbb{F}$, we can think of what it means to do plane geometry over $\mathbb{F}$: our "plane" is the set $\mathbb{F}\times\mathbb{F}$ (whose elements are points), and lines are subsets

$$\ell = \{(x,y) \in \mathbb{F}\times\mathbb{F} : ax+by+c=0 \text{ for fixed } a,b,c\in\mathbb{F}\}$$

   Prove that over any field $\mathbb{F}$, two points define a unique line.

[1] If you don't know the formal definition of a field, come ask me!

(b) Conclude that the proposition we proved in class is true when working over $\mathbb{F}$, and that the bounds in Exercise 2 are true over $\mathbb{F}$ as well.

(c) There exist finite fields! The simplest examples are the integers modulo p for some prime number p, though there are other finite fields as well. Prove that over any finite field, the bounds in Exercise 2 are tight, in the sense that we can find some $C \subseteq \mathbb{F} \times \mathbb{F}$ and some collection L of lines in $\mathbb{F} \times \mathbb{F}$ such that

$$I(C, L) = O(mn^{1/2} + n) = O(m^{1/2}n + m)$$

This means that the Szmerédi-Trotter Theorem really captures some special property of the Euclidean plane $\mathbb{R}^2$.