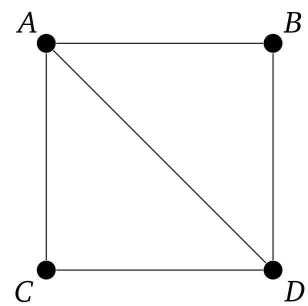
All problems are optional, but in my opinion fun to think about. A problem marked with a * is hard, and a problem marked with ** is very hard—in particular, I don't know how to solve it fully.

1. Believe it or not, the obnoxious infinite series technique we used to calculate hitting times in class can actually be useful. Specifically, consider the following graph:



   (a) Calculate the hitting itme $H(A, D)$ using the second technique we saw in class (recursively using the neighbor-averaging property of random walks).

   (b) Using the infinite series technique we saw in class, find an infinite series whose sum is $H(A, D)$.

   (c) My copy of Mathematica can't sum this infinite series, and neither can Wolfram Alpha (at least when I try). However, you know that its sum equals $H(A, D)$, and you already know what that is from part (a). Conclude that you know how to sum an infinite series that Mathematica doesn't know how to sum.

   (d)** I generated this example sort of by accident. In particular, I know of no general method of starting from some infinite series and creating a graph whose hitting time is calculated by that series. Can you come up with such a method? What sort of conditions are satisfied by series that calculate hitting times? I have a few partial results, but would love some more.

2. In class, we proved that there are no non-constant harmonic functions on finite connected graphs, and we saw that both the non-constant and the connected restrictions are necessary. Here, we will see that the finiteness restriction is necessary too.

   (a) Construct an infinite graph $G = (V, E)$ and a function $f : V \to \mathbb{R}$ such that f is harmonic on every vertex of G, but f is not constant.

   (b)* In all likelihood, the function f that you constructed in part (a) is not a bounded function. This time, construct a new graph G and a new function f such that f is non-constant, harmonic on all of G, and bounded.

   Note: If you want to learn more about examples like this, stop taking this class and go take Fedya's Ponzi Schemes class instead.

1. Prove that if f_1 and f_2 are both harmonic at some vertex $v \in V$, and $\alpha, \beta \in \mathbb{R}$ are any real numbers, then $\alpha f_1 + \beta f_2$ is also harmonic at v. This implies that the set of functions harmonic at v is a vector space. Note that this is more general than the result we saw in class, namely that $f_1 - f_2$ is harmonic.

2. In class, we saw that for any graph $G = (V, E)$ and for any $B \subseteq V$ and for any function $f_0 : B \to \mathbb{R}$, there is a unique harmonic extension of f_0 to all of V. Now, we will focus on the case when $|B| = 2$, say $B = \{s, t\}$.

   (a) Using Exercise 1, prove that we might as well suppose that $f_0(s) = 0$ and $f_0(t) = 1$.

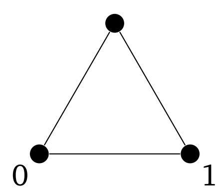
   (b) Recall that in class, we defined the harmonic extension of f_0 to be

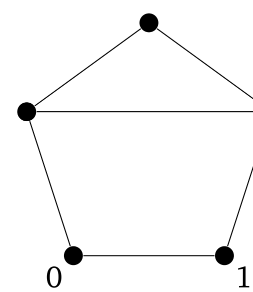
   $$f(v) = \sum_{b \in B} f_0(b)\,\mathrm{Prob}(b \text{ is the first vertex in } B \text{ that a random walk from } v \text{ reaches})$$

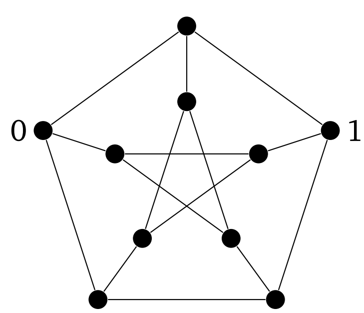
   Write this more concisely for the case when $B = \{s, t\}, f_0(s) = 0, f_0(t) = 1$.

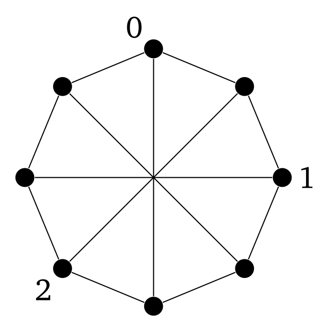
   (c) Convince yourself that your more concise form from part (b) is harmonic on $V \setminus B$. This follows from our proof in class, but is simpler to think about in this special case.

3. Calculate the unique harmonic extensions on the following graphs, where the vertices with numbers next to them are the vertices in B and the numbers denote the value of f_0 at that vertex.

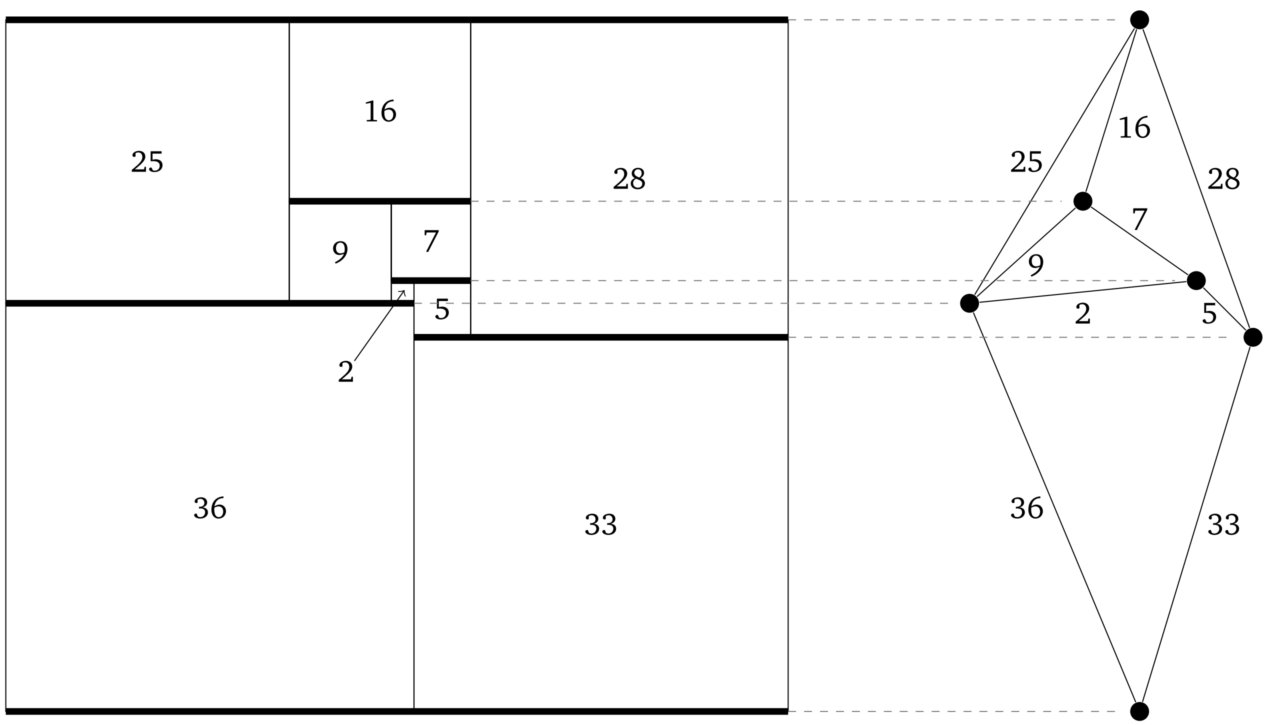




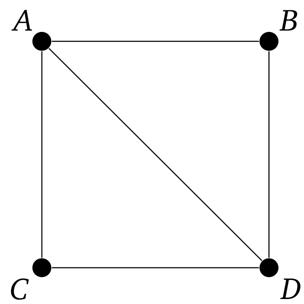




Here is the square tiling example from class, drawn much better than on the board:



1. Using our new method to compute hitting times (namely, nailing the target vertex to the wall, attaching a weight equal to the degree to every other vertex, and turning edges into rubber bands), re-compute the hitting time you found on the first homework, i.e. $H(A, D)$ in the following graph:



2. In class, I stated that two notions were equivalent: (1) there is no pair of vertices whose removal disconnects the graph, and (2) for any set of vertices S, there are at least 3 edges leaving S. As Milan pointed out to me, these are emphatically **not equivalent.**

   Prove that (1) implies (2), and construct an example satisfying (2) but not (1). Only property (1) is called 3-connectivity.

3.* Prove that if G is planar and we add edges (without duplicating any edge) so that every face is a triangle, then the new graph we get is planar and 3-connected.

4.* In this problem, we will explore one of the harmonic connections that I stated but didn't prove in class. This problem requires some background in probability, so feel free to skip it if you haven't seen things like expectation or conditional probability.

   (a) Define the *commute time* between vertices s, t in a graph G to be the expected length of a random walk that starts at s, reaches t, and then returns to s; denote the commute time by C_{st}. Prove that

   $$C_{st} = H(s, t) + H(t, s)$$

   (b) For a vertex $v \in V$, let

   $$f(v) = \text{Prob(a random walk from } v \text{ reaches } t \text{ before it reaches } s)$$

   Prove that $f(s) = 0, f(t) = 1$, and f is harmonic on $V \setminus \{s, t\}$.

   (c)* Let S denote the random variable equal to the first time that a random walk starting at s returns to s. Let $\mathbb{E}(S)$ denote the expected value of S. Then it turns out that

   $$\mathbb{E}(S) = \frac{2|E|}{\deg(s)}$$

   This is a hard fact from the theory of random walks, but try to prove it if you're interested.

(d) Let T denote the random variable measuring the first time the walk returns to s after visiting t. Then by definition, $\mathbb{E}(T) = C_{st}$. In addition, $S \leq T$. Prove that

$$\text{Prob}(S = T) = \frac{1}{\deg(s)} \sum_{u \in N(s)} f(u)$$

(e) Prove that

$$\mathbb{E}(T - S \mid S < T) = \mathbb{E}(T)$$

where the bar | denotes conditional probability.

(f) Put everything together to get that

$$C_{st} = \frac{2|E|}{\sum_{u \in N(s)} f(u)}$$

which is the connection to rubber bands (and everything else) that I stated in class.