1. An *independent set* in a graph G is a subset $I \subseteq V(G)$ that contains no edges, i.e. that $x \nsim y$ for all $x, y \in I$. The *independence number* of G, denoted $\alpha(G)$, is the size of the largest independent set in G. Prove that for any simple graph G,

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

2. Recall that the Kneser graph $\mathrm{KG}(n,k)$ is defined, for $n \geq k$, as the graph whose vertices are all k-element subsets of $[n]$, and where subsets S, T are adjacent if $S \cap T = \varnothing$.

   (a) Draw $\mathrm{KG}(6,2)$.

   (b) Describe $\mathrm{KG}(n,k)$ if $n < 2k$.
   Because of the result of this exercise, from now on we assume that $n \geq 2k$.

   (c) Describe $\mathrm{KG}(2k,k)$.

   (d) Describe and $\mathrm{KG}(n,0)$ and $\mathrm{KG}(n,1)$.

   (e) How many edges does $\mathrm{KG}(n,k)$ have?
   Hint: Think about the degree of each vertex, namely how many edges each vertex is incident to.

   (f) Recall that $\alpha(G)$ denotes the size of the largest independent set in G. Show that $\alpha(\mathrm{KG}(n,k)) \geq \binom{n-1}{k-1}$.
   Hint: What does it mean for a set of vertices in $\mathrm{KG}(n,k)$ to be an independent set? What's a good way to find a large set with this property?

   $\star$ (g) Prove that $\alpha(\mathrm{KG}(n,k)) = \binom{n-1}{k-1}$.

   ✥ (h) Prove that $\chi(\mathrm{KG}(n,k)) \leq n - 2k + 2$.

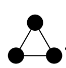
   ✥$\star\star$ (i) Prove that $\chi(\mathrm{KG}(n,k)) = n - 2k + 2$.

3. Recall that the chromatic number $\chi(G)$ is defined as the minimum c for which there exists a proper coloring $\varphi : V(G) \to [c]$. Similarly, we can try to define the *co-chromatic number* $\kappa\chi(G)$ to be the minimum c for which there exist $\varphi_1, \varphi_2 : V(G) \to [c]$ for which φ_1, φ_2 are co-proper colorings.

   (a) Prove that $\kappa\chi(G) \leq 2$ for all graphs G.

   (b) For which graphs G does $\kappa\chi(G) = 1$?

   (c) Use the two facts you just proved to determine $\kappa\chi(G)$ for every G, and conclude that the co-chromatic number is not a particularly interesting concept to study.

4. Let K_3 denote the triangle . Draw $\mathcal{E}_2(K_3)$.

✥ 5. Prove, as mentioned in class, that a pair of co-proper c-colorings of G is the same as a proper c-coloring of $G \times K_2$, where K_2 is the graph with two vertices and one edge.

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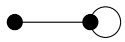
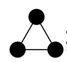
$\star$6. Suppose (G, H) is a counterexample to Hedetniemi's conjecture, i.e. suppose that $\chi(G) \geq c+1, \chi(H) \geq c+1$, but $\chi(G \times H) \leq c$. Show that $(G, \mathcal{E}_c(G))$ is also a counterexample to Hedetniemi's conjecture. Because of this, it makes sense to only focus on exponential graphs, as we are indeed doing in this class.
Hint: Suppose $\Phi : V(G \times H) \to [c]$ is a proper c-coloring. Use Φ to construct a map $\varphi_h : V(G) \to [c]$ for every $h \in V(H)$. This yields a map $V(H) \to V(\mathcal{E}_c(G))$; how does this map interact with the graph structures of H and $\mathcal{E}_c(G)$?

✣7. Given two graphs G, H, another important graph product is their *Cartesian product* $G \square H$, defined as follows. The vertex set of $G \square H$ is $V(G) \times V(H)$, and two pairs (g_1, h_1) and (g_2, h_2) are adjacent if

$$g_1 = g_2 \text{ and } h_1 \sim h_2 \qquad \text{or} \qquad g_1 \sim g_2 \text{ and } h_1 = h_2.$$

(a) Let $G =$  and $H =$ ; in class, we drew $G \times H$. Draw $G \square H$.

(b) Prove that $\chi(G \square H) \geq \max\{\chi(G), \chi(H)\}$ for any graphs G, H.

(c) Prove the "Cartesian Hedetniemi conjecture": for all graphs G, H,

$$\chi(G \square H) = \max\{\chi(G), \chi(H)\}.$$

(d) Can you figure out why the notation $\square$ and $\times$ is used for the Cartesian product and tensor product, respectively? If I tell you that there's another graph product denoted $\boxtimes$, then can you guess how $G \boxtimes H$ is defined?

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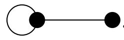
1. In this exercise, we'll go through most of what we did today in class for an explicit example. Let $G =$ .

   (a) Draw $\mathcal{E}_3(G)$.
   Hint: I think that $\mathcal{E}_3(G)$ has 9 vertices and 12 edges.

   (b) Find the three constant maps $\varphi_1, \varphi_2, \varphi_3$ as vertices of $\mathcal{E}_3(G)$. Verify that they are all pairwise adjacent.

   (c) Recall that in class, for a vertex v of G and a color $b \in [c]$, we defined a set $S_{v,b}$ to consist of all maps φ that map v to b, i.e. $S_{v,b} = \{\varphi : V(G) \to [c] \mid \varphi(v) = b\}$. Verify that if x is the vertex of G with a loop, then $S_{x,b}$ is an independent set in $\mathcal{E}_3(G)$ for all $1 \leq b \leq 3$, whereas if y is the vertex of G without a loop, then $S_{y,b}$ need not be an independent set.

   (d) Find at least two proper 3-colorings of $\mathcal{E}_3(G)$. In the first coloring, have the color classes be the sets $S_{x,1}, S_{x,2}, S_{x,3}$. In the second coloring, use a different partition of the vertices into color classes.

   (e) Check that in both of the colorings from the previous part, you can rename the color classes so that these colorings are suited.

   $\star$ (f) Consider going through everything we did in class today and making sure that every step makes sense, using $\mathcal{E}_3(G)$ as an explicit example to keep track of.

✥$\star$ 2. In this exercise, we'll prove the famous five-color theorem, which says that if G is a planar graph (meaning it can be drawn in the plane without crossing edges), then $\chi(G) \leq 5$. As you may know, the stronger four-color theorem is also true, but it is **much** harder to prove.

   $\star$ (a) Prove Euler's formula, which says that if G is a planar graph with V vertices, E edges, and which defines F faces (including the infinite face), then $V - E + F = 2$.
   Hint: First, prove this if G is a tree, by induction on the number of vertices. Then use induction on the number of edges.

   $\star$ (b) Prove that if a planar graph has $V \geq 3$ vertices and E edges, then $E \leq 3V - 6$.
   Hint: Use Euler's formula.

   (c) Using the previous part, prove that if G is a planar graph, then it has a vertex of degree at most 5.

   (d) Deduce the six-color theorem: if G is a planar graph, then $\chi(G) \leq 6$.

   (e) Suppose you have a proper coloring of any graph. A *Kempe chain* is a connected subgraph using only two colors. Prove that if you take any Kempe chain and swap the colors of all vertices inside it (and don't change the color of any other vertex), then you will get a new proper coloring.

   $\star$ (f) Prove the five-color theorem, by induction on the number of vertices. The key step is when a vertex of degree ≤ 5 needs to be colored, and can't: in that case, find an appropriate pair of Kempe chains to color-swap.

$\star$ *means a problem is hard;* ✥ *means a problem is not directly related to the content of the course. Feel free to ignore these problems, but you should try all problems without a symbol.*

✥ 3. To solve this problem, you need to know some (or much) category theory.

(a) A *graph homomorphism* from G to H is a map $\varphi : V(G) \to V(H)$ such that if $x \sim y$ in G, then $\varphi(x) \sim \varphi(y)$ in H. Verify that this yields a category whose objects are all graphs and whose arrows are all graph homomorphisms.

(b) Prove that G can be properly c-colored if and only if there exists some homomorphism $G \to K_c$. Thus, $\chi(G)$ is the minimum c such that there is a homomorphism $G \to K_c$.

(c) Prove that in the category of graphs, the categorical product is given by the tensor product.

(d) Prove that in the category of graphs, the coproduct is given by the disjoint union (i.e. the graph $G \sqcup H$ formed by putting G and H next to each other and placing no edges between them).

(e) Prove that $\chi(G \sqcup H) = \max\{\chi(G), \chi(H)\}$.

(f) By the above, disjoint union is dual to tensor product, and maximum is dual to minimum. Thus, dualizing Hedetniemi's conjecture yields the statement $\chi(G \sqcup H) = \max\{\chi(G), \chi(H)\}$. Moreover, if we dualize everything in a true statement, we get another true statement, so we've just proved Hedetniemi's conjecture. What's wrong with this argument?

$\star$ (g) Find the true statement that is actually dual to $\chi(G \sqcup H) = \max\{\chi(G), \chi(H)\}$.

✥ (h) Given two graphs G, H, define a new graph $\hom(G, H)$ as follows. The vertices of $\hom(G, H)$ are all functions $\varphi : V(G) \to V(H)$, and $\varphi_1 \sim \varphi_2$ in $\hom(G, H)$, if whenever $x \sim y$ in G, then $\varphi_1(x) \sim \varphi_2(y)$ in H. Prove that $\hom(G, K_c) = \mathcal{E}_c(G)$ for all G, c.

✥ (i) Prove that $\times$ and hom are adjoint. Namely, prove that there is a natural bijection between the set of all homomorphisms $G \times H \to K$ and the set of all homomorphisms $H \to \hom(G, K)$, for all graphs G, H, K. This implies that the category of graphs is *Cartesian closed.*

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1. Go through the proof we did today in class (at least of the first step: if $I(u, b)$ is large, then b is u-robust). Despite a lot of trying, I haven't been able to come up with an exercise that makes it more understandable, so unfortunately your exercise is just to go through it and come talk to me if you're confused.

2. Given two vertices x, y in a graph, their distance $\operatorname{dist}(x, y)$ is defined to be the number of edges in the shortest path connecting them.

   (a) Suppose G is a connected graph. Verify that the distance is a well-defined function, and prove that it satisfies the following properties:

   - $\operatorname{dist}(x, y) \geq 0$ for all x, y, and $\operatorname{dist}(x, y) = 0$ if and only if $x = y$
   - $\operatorname{dist}(x, y) = \operatorname{dist}(y, x)$ for all x, y
   - For every x, y, z,
   $$\operatorname{dist}(x, z) \leq \operatorname{dist}(x, y) + \operatorname{dist}(y, z).$$
   This is called the *triangle inequality.*

   Any function satisfying these three properties is called a *metric.*

   (b) Suppose that G is connected and that every cycle in G has length at least $2k$, for some integer k. Suppose that $x \sim y$ and that $\operatorname{dist}(x, v) < k, \operatorname{dist}(y, v) < k$, for some vertex v. Prove that $\operatorname{dist}(x, v)$ and $\operatorname{dist}(y, v)$ have different parities (i.e. one is even and the other is odd).

⊕ 3. A b-fold proper coloring of a graph G is an assignment of a set of b colors to every vertex of a graph such that adjacent vertices get disjoint color sets. The *b-fold chromatic number* of G, denoted $\chi(G; b)$ is the minimum total number of colors needed for a proper b-fold coloring of G. The *fractional chromatic number* of G, denoted $\chi_f(G)$, is defined by
$$\chi_f(G) = \min_{b \in \mathbb{N}} \frac{\chi(G; b)}{b}.$$

   (a) Prove that $\chi_f(G) \leq \chi(G)$.

   (b) Prove that $\chi_f(G) \geq |V(G)|/\alpha(G)$.

   (c) Prove that for $n \geq 2k$,
   $$\chi_f(\mathrm{KG}(n, k)) = \frac{n}{k}.$$

   ⋆ (d) This part requires you to know some analysis. Prove that
   $$\chi_f(G) = \lim_{b \to \infty} \frac{\chi(G; b)}{b}.$$

⊕ 4. For this problem, you'll need to know some linear algebra. Given an $m \times m$ matrix A and an $n \times n$ matrix B, their *tensor product* (also called the *Kronecker product*) is the $(mn) \times (mn)$ matrix $A \otimes B$ defined as follows. We split the mn rows and columns into n^2 blocks of size $m \times m$. Then in the (i, j) position of the (k, ℓ) block, put the value $a_{i,j} b_{k,\ell}$.

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(a) If G is a graph, its adjacency matrix $A(G)$ is the matrix whose rows and columns are indexed by the vertices of G, and the (u, v) entry of $A(G)$ is 1 if $u \sim v$ and 0 otherwise. Prove that for any graphs G, H,

$$A(G \times H) = A(G) \otimes A(H),$$

which is why $G \times H$ is called the tensor product of G and H.

$\star$ (b) Prove that $A \otimes B \neq B \otimes A$ in general, but that they are equal up to re-ordering the rows and columns. On the other hand, prove that $\otimes$ is associative.

(c) Prove that if A, C are $m \times m$ matrices and B, D are $n \times n$ matrices, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

$\oplus\star$ (d) To do this exercise, you need to know what the tensor product of vector spaces is. Show that if V, W are vector spaces and $T : V \to V, S : W \to W$ are linear transformations, then we can define a natural linear transformation $(T \otimes S) : V \otimes W \to V \otimes W$. Prove that if A is the matrix representing T in some basis, and B represents S in some basis, then $A \otimes B$ represents $T \otimes S$ in an appropriate basis of $V \otimes W$.

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1. Prove that for all simple graphs G, H,
$$(G \boxtimes H)^\circ = G^\circ \times H^\circ.$$
In other words, the strong product can be gotten by adding a loop to every vertex, taking the tensor product, and then removing all loops.

2. (a) Prove that for every G and every positive integer q, $\alpha(G \boxtimes K_q) = \alpha(G)$.
   (b) Prove that
   $$\chi(G \boxtimes K_q) \geq q\frac{|V(G)|}{\alpha(G)}.$$
   (c) Prove that $\chi(G \boxtimes H) \leq \chi(G)\chi(H)$ for any simple graphs G, H.
   ✥⋆(d) Does equality hold in general in either of the inequalities in the previous two parts?

✥⋆3. (a) Since $\alpha(K_q) = 1$, 2(a) implies that $\alpha(G \boxtimes K_q) = \alpha(G)\alpha(K_q)$. Does this hold in general? In other words, is it the case that $\alpha(G \boxtimes H) = \alpha(G)\alpha(H)$ for all G, H?
   (b) Since $|V(K_q)| = q\alpha(K_q)$, 2(b) implies that
   $$\chi(G \boxtimes K_q) \geq \frac{|V(G)|}{\alpha(G)}\frac{|V(K_q)|}{\alpha(K_q)}.$$
   Does this hold in general? In other words, does
   $$\chi(G \boxtimes H) \geq \frac{|V(G)|}{\alpha(G)}\frac{|V(H)|}{\alpha(H)}$$
   hold for all G, H?

4. Recall that the *girth* of a graph is defined to be the length of its shortest cycle, and ∞ if the graph has no cycles.
   (a) Prove that if $\text{girth}(G) = \infty$, then $\chi(G) \leq 2$.
   (b) Prove that if $\chi(G) \leq 2$, then $\text{girth}(G) \geq 4$.
   (c) The previous two parts suggest that it's hard to find a graph such that both its girth and its chromatic number are large. Try to construct a graph G with $\text{girth}(G) \geq 5$ and $\chi(G) \geq 3$. How about $\text{girth}(G) \geq 4$ and $\chi(G) \geq 4$?

✥5. A very famous construction, due to Mycielski, shows that there exist graphs of girth ≥ 4 and arbitrarily large chromatic number. Let G be an arbitrary graph with vertex set $x_1, \ldots, x_n$. We construct a new graph $\mathcal{M}(G)$ whose vertex set is $x_1, \ldots, x_n, y_1, \ldots, y_n, z$, as follows. First, among the x vertices, the edges are the same as they were in G, i.e. $x_i \sim x_j$ in $\mathcal{M}(G)$ if and only if $x_i \sim x_j$ in G. Next, for each $1 \leq i \leq n$, y_i is adjacent to the same vertices as x_i, i.e. $y_i \sim x_j$ if and only if $x_i \sim x_j$. Finally, z is adjacent to every y_i, and no other vertices.

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(a) Prove that if $\mathrm{girth}(G) \geq 4$, then $\mathrm{girth}(\mathcal{M}(G)) \geq 4$ as well.

$\star$ (b) Prove that $\chi(\mathcal{M}(G)) = \chi(G) + 1$.

(c) Set G_2 to be the graph with two vertices and a single edge connecting them, and inductively $G_i = \mathcal{M}(G_{i-1})$ for $i > 2$. Conclude that $\mathrm{girth}(G_i) \geq 4$ and $\chi(G_i) = i$ for all i.

$\star\star$ (d) Can you find a way to produce a graph with $\mathrm{girth}(G) \geq a$ and $\chi(G) \geq b$ for every a, b?

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