High-Girth Matrices and Polarization

Emmanuel Abbe Yuval Wigderson

Princeton University

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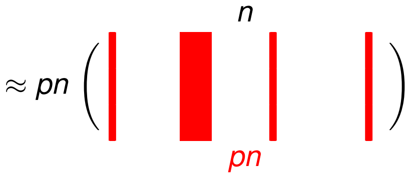

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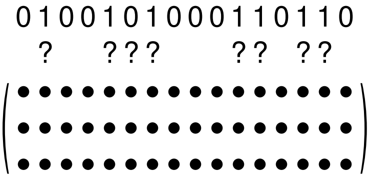

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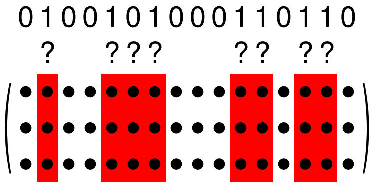

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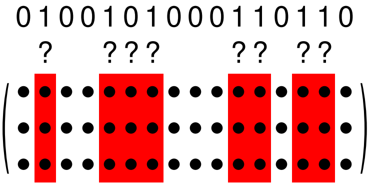

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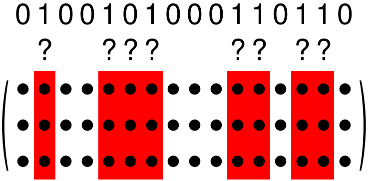

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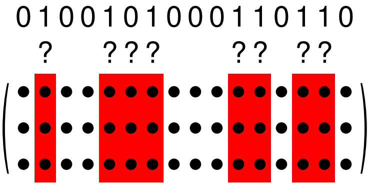

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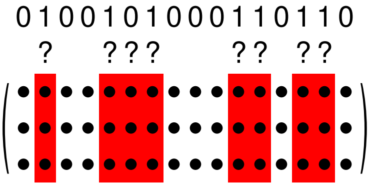

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Polar codes and Reed-Muller codes are high-girth submatrices of G_n. How about other submatrices? [Arıkan, Kumar-Pfister, Kudekar et al.]

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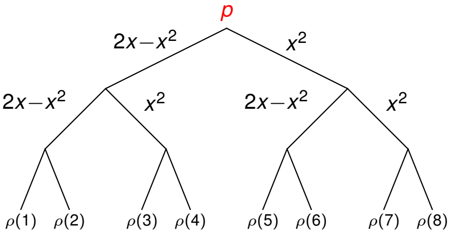

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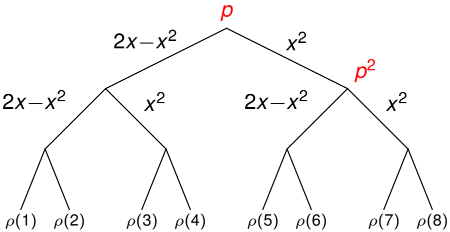

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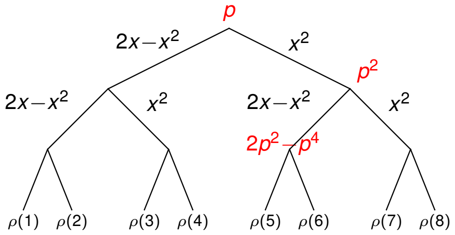

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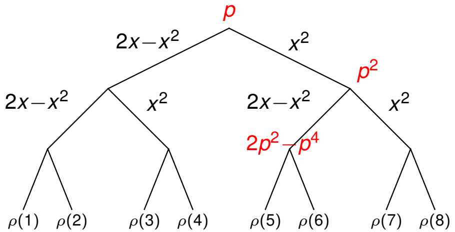

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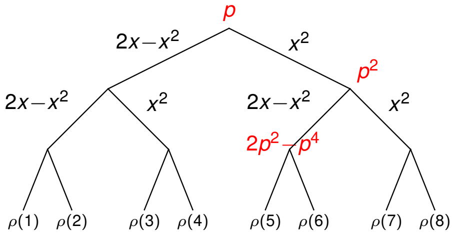

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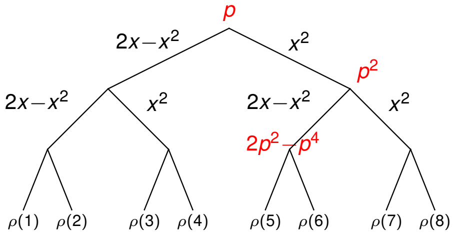

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Note 3: An upper bound for the Z process on any channel.

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4. Working over $\mathbb{R}$, these COR matrices are binary matrices with good Sparse Recovery properties (can distinguish most pairs of sparse patterns).

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3. What about error channels over larger fields?
   AWGN or other continuous channels?
   Polarization over other combinatorial objects?

Thank you!