Spectral bounds on the independence number

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Joint with Matthew Kwan

Outline

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General problem: Understand the space of all WAMs of G, and optimize some quantity over this space.

The ratio bound

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Theorem (Hoffman (unpublished))

Let G be a regular n-vertex graph with adjacency matrix A. Then

$$\alpha(G) \leqslant \left| \frac{\lambda_{\min}(A)}{\lambda_{\min}(A) - \lambda_{\max}(A)} \right| n.$$

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Even for general graphs, this optimization is a semidefinite program, so the optimum is efficiently computable.

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No! The Paley graph P_{17} has $\alpha(P_{17}) = 3$ but $n_+(W) \geqslant 4$ for every weighted adjacency matrix W.

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The proof involves a lot of casework and is very specific to P_{17}.

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This yields an infinite family of examples for Godsil's question.

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Theorem (Kwan–W. (2024))

If G is C_4-free, then $n_+(W) \geqslant 0.232n$ for every WAM W of G.

If G is the polarity graph of a projective plane, then G is C_4-free and the ratio bound proves $\alpha(G) \leqslant \alpha_q(G) = O(n^{3/4})$.

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The limits of the inertia bound

Proof sketch

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- Note that

$$\mathbb{E}[X] = \frac{1}{n}\operatorname{tr}(W) = 0.$$

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Lemma

If X is a RV with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$, and $\mathbb{E}[X^4] \leqslant 2$, then

$$\Pr(X \geqslant 0) \geqslant \sqrt{3} - \tfrac{3}{2} \approx 0.232.$$

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Remarkably, we will be able to reduce to this case.

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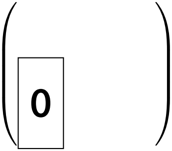

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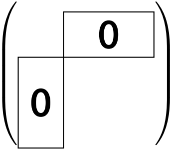

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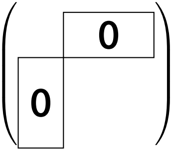

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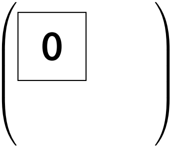

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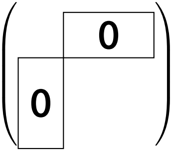

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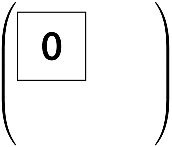

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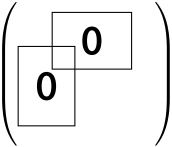

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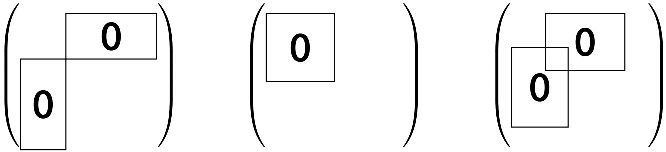

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In any case, we are done by induction + Cauchy interlacing.

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$$n_+(W) = n_+(W') = n \cdot \Pr(X \geqslant 0) \geqslant 0.232n. \qquad \square$$

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Question

What's the complexity of computing the best possible inertia bound?

Thank you!