Spectral bounds on the independence number

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Outline

Spectral graph theory

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General problem: Understand the space of all WAMs of G, and optimize some quantity over this space.

The ratio bound

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Theorem (Hoffman (unpublished))

Let G be a regular n-vertex graph with adjacency matrix A. Then

$$\alpha(G) \leq \left| \frac{\lambda_{\min}(A)}{\lambda_{\min}(A) - \lambda_{\max}(A)} \right| n.$$

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Even for general graphs, this optimization is a semidefinite program, so the optimum is efficiently computable.

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The independent set S yields an all-zeroes principal minor M of A. Cauchy's interlacing formula implies $n_{\geq 0}(A) \geq n_{\geq 0}(M) = \alpha(G)$. □

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The limits of the inertia bound

Proof sketch

Godsil's question

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The proof involves a lot of casework and is very specific to P_{17}.

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For infinitely many n, there exists an n-vertex graph G with

$$\alpha(G) = O\left(n^{0.9999}\right) \qquad \text{and} \qquad \alpha_q(G) = \Omega\left(\frac{n}{\log n}\right).$$

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This yields an infinite family of examples for Godsil's question.

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Theorem (Kwan–W. (2023+))

If G *is* C_4*-free, then* $n_{\geq 0}(W) \geq 0.232n$ *for every WAM* W *of* G*.*

If G is the polarity graph of a projective plane, then G is C_4-free and the ratio bound proves $\alpha(G) = O(n^{3/4})$.

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  Also, by rescaling W, we may assume $\mathbb{E}[X^2] = 1$.

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- Controlling the first two moments is not enough to learn about $\Pr(X \geq 0)$.

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- Fix a WAM W of G, with eigenvalues $\lambda_1, \ldots, \lambda_n$.
- Let X be the random variable taking value λ_i with probability $1/n$, for $i \in [n]$. We want to prove a lower bound on $\Pr(X \geq 0)$.
- Note that
$$\mathbb{E}[X] = \frac{1}{n}\operatorname{tr}(W) = 0.$$
Also, by rescaling W, we may assume $\mathbb{E}[X^2] = 1$.
- Controlling the first two moments is not enough to learn about $\Pr(X \geq 0)$. However:

Lemma

If X is a RV with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] = 1$, and $\mathbb{E}[X^4] \leq 2$, then

$$\Pr(X \geq 0) \geq \sqrt{3} - \tfrac{3}{2} \approx 0.232.$$

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Remarkably, we will be able to reduce to this case.

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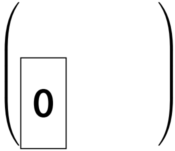

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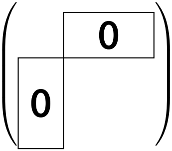

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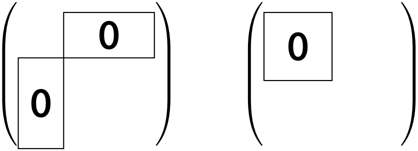

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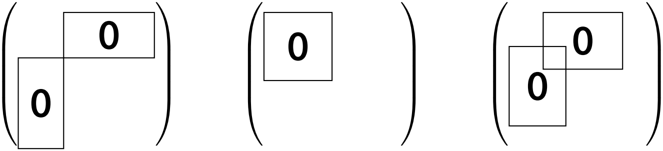

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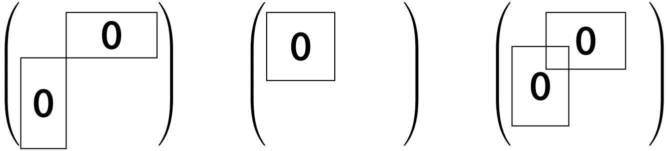

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In any case, we are done by induction + Cauchy interlacing.

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$$n_{\geq 0}(W) = n_{\geq 0}(W') = n \cdot \Pr(X \geq 0) \geq 0.232n.$$

□

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Let $G \sim \mathbb{G}(n, \frac{1}{2})$*. With probability* $1 - o(1)$*, every WAM of* G *satisfies*

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For all k, *there exists* G *with* $\alpha(G) = 2$ *but* $n_{\geq 0}(W) \geq k$ *for any WAM.*

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Question

Is it decidable to compute the best possible inertia bound?

Thank you!