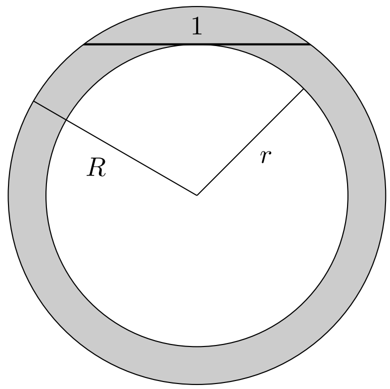
1 Recall that one suggestion in class for a small-area Kakeya set is an *annulus*, namely the area between two concentric circles. Prove that if you have an annulus of inner radius $r$ and outer radius $R$ such that it can fit a unit line segment, then its area is $\frac{\pi}{4}$, the same as the circle of radius $\frac{1}{2}$.




2 In class, our definition of a Kakeya set was not entirely rigorous, since we didn't really say what it means to continuously rotate a bedframe inside $K$. Write down a formal definition. [This exercise requires you to know some calculus or analysis.]

3* Formally prove that every Kakeya set in $\mathbb{R}^2$ has positive area.

4 (a) Recall that a Besicovitch set in $\mathbb{R}^2$ is a set that contains a unit line segment in every direction. Similarly, define a Besicovitch set in $\mathbb{R}^n$ for any $n$.

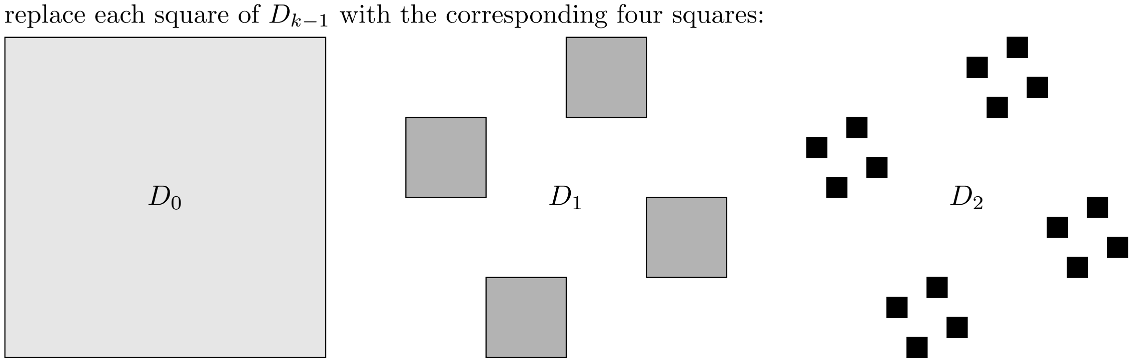
(b) Recall that $Z$ is an area-zero set in $\mathbb{R}^2$ if for every $\varepsilon > 0$, there is an open set $U_\varepsilon \supset Z$ with area$(U_\varepsilon) \leq \varepsilon$. Similarly, define a volume-zero set in $\mathbb{R}^n$ for any $n$.

(c) Prove that there exists a volume-zero Besicovitch set $B \subset \mathbb{R}^n$ for all $n \geq 2$. For this problem, you can assume the fact that we'll prove tomorrow, that there is an area-zero Besicovitch set in $\mathbb{R}^2$.

(d) Prove that there is no volume-zero (or rather length-zero) Besicovitch set in $\mathbb{R}^1$.

5 In this problem, we will define the *Cantor dust*, which is another way to construct area-zero Besicovitch sets in $\mathbb{R}^2$. This is not at all the strategy we will employ tomorrow.

(a) Let $D_0$ be the unit square in $\mathbb{R}^2$, with corners at $(0,0), (0,1), (1,0), (1,1)$. Then, let $D_1$ be the subset of $D_0$ consisting of four squares of side length $\frac{1}{4}$, with lower-left corners at $(0, \frac{1}{2}), (\frac{1}{4}, 0), (\frac{3}{4}, \frac{1}{4})$, and $(\frac{1}{2}, \frac{3}{4})$. We now iterate this construction, at each step letting $D_k$ be what we get when we replace each square of $D_{k-1}$ with the corresponding four squares:




Finally, let $D = \bigcap_{k=1}^{\infty} D_k$; $D$ is called the Cantor dust. Prove that $D$ is a zero-area set.

(b) Prove that when we project $D$ onto the $x$-axis, we get the whole interval $[0,1]$. In other words, prove that for all $a \in [0,1]$, there is some $b \in [0,1]$ such that $(a,b) \in D$. Similarly, prove that $D$

has a full projection onto the $y$-axis; on the other hand, prove that its projection onto any other line in the plane has length zero.

(c) Let

$$E = \bigcup_{(a,b)\in D} \ell_{a,b} \qquad \text{where} \qquad \ell_{a,b} = \{(x,y) : y = ax + b\}$$

This is called the *dual construction*; $E$ consists of all lines that are parametrized by points of $D$. Try to picture what $E$ looks like!

(d) Conclude that $E$ contains a full line (not just a line segment) in every direction between $0°$ and $45°$. Therefore, conclude that $E$, along with seven additional rotated copies of it, is a Besicovitch set in $\mathbb{R}^2$.

(e)* Using (5b), prove that $E$ has zero area. [This is very very hard, and is in fact more or less how Besicovitch originally proved his theorem. To prove it, he developed the so-called theory of *irregular 1-sets*.]

(f) The only property of $D$ that we used to get a Besicovitch set from it is that it has a full projection on to the $x$-axis. Many other sets have this property; can you find another one that seems to give an area-zero Besicovitch set?

1 In this problem, we will explore a heuristic for determining the dimension of self-similar sets (often called fractals). A self-similar set is a set $S \subset \mathbb{R}^n$ such that $S$ can be decomposed as $k$ copies of itself, each scaled down by a factor $r$.

(a) Check that the Sierpinski gasket is self-similar, with $k = 3, r = \frac{1}{2}$.

(b) Check that the Cantor dust from yesterday's homework is self-similar. What are $k$ and $r$ here?

(c) Some non-fractals are also self-similar. Check that a square and an equilateral triangle are both self-similar, and find $k$ and $r$ for them.

(d) We know that if we scale a 2-dimensional shape by a factor $r$, then its area changes by a factor $r^2$, and if we scale a 3-dimensional shape by a factor $r$, then its volume changes by a factor $r^3$. This suggests that if a set has "dimension" $d$ and "size" $s$, then scaling it by $r$ will change its "size" by a factor of $r^d$.

Therefore, if $S$ is self-similar of "size" $s$ and "dimension" $d$, then by decomposing it into $k$ smaller copies of itself, we expect that

$$s = k \cdot s \cdot r^d$$

since the left-hand side is the "size" of $S$, and the right-hand side is the "size" of $k$ copies of $S$, scaled down by $r$. Use this to get a formula for $d$ in terms of $k$ and $r$.

(e) Using the formula from 1d, compute what the dimension of the Sierpinski gasket should be, and check that this agrees with what we found in class.

(f) Again using this formula, compute what you expect the dimension of the Cantor dust to be. Conclude that even though fractals *can* have a non-integer dimension, they need not.

(g) Check that this formula also gives the right value for the dimension of the square and the equilateral triangle.

(h) If you're so inclined, feel free to find other self-similar sets and compute their heuristic dimension.

2 Calculate the Minkowski dimension of the Cantor dust, the square, and the triangle, and check that these agree with what you found in Problem 1.

3 Suppose we partition $\mathbb{R}^n$ into cubes of side length $\varepsilon$ by drawing a very fine lattice; let $N'(S, \varepsilon)$ be the number of these cubes that intersect some bounded set $S$. Prove that we can equivalently define the Minkowski dimension of $S$ using $N'$, namely that

$$\dim_M S = \lim_{\varepsilon \to 0} \frac{\log N'(S, \varepsilon)}{-\log \varepsilon}$$

In fact, just about every similar notion you could think of gives an equivalent definition of Minkowski dimension—for instance, covering $S$ by cubes, covering $S$ by tetrahedra, partitioning $\mathbb{R}^n$ into rectangular prisms, etc.

4* Construct a bounded set $S \subset \mathbb{R}^n$ (for some $n$) for which the limit

$$\lim_{\varepsilon \to 0} \frac{\log N(S, \varepsilon)}{-\log \varepsilon}$$

does not exist. Thus, $S$ does not have a well-defined Minkowski dimension.

1 Prove that any two points in $\mathbb{F}_q^n$ have a unique line going through them, and that any two lines in $\mathbb{F}_q^n$ intersect in at most one point. If you're so inclined, check that various other properties of points and lines that we're used to from Euclidean geometry also translate over to the finite field case. On the other hand, remember that many notions simply can't be transferred in this way; in particular, any facts about "length" or "angle" have no analogue over finite fields.

2 Find a polynomial $P(x)$ over $\mathbb{F}_q$ that is non-zero but that vanishes on all of $\mathbb{F}_q$. If you haven't seen other finite fields, only prove this for the integers mod a prime.

3 In this problem, you will prove that every Besicovitch set in $\mathbb{R}^n$ has Minkowski dimension at least $n/2$. This is certainly far from the full Kakeya Conjecture, but at least shows that Besicovitch sets can't be *too* small.

(a) Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be bounded subsets with well-defined Minkowski dimension. Prove that

$$\dim_M(X \times Y) = \dim_M(X) + \dim_M(Y)$$

where $X \times Y \subset \mathbb{R}^{n+m}$ is the standard Cartesian product.

(b) Prove that if $X \subset \mathbb{R}^n$ is a bounded subset with well-defined Minkowski dimension and if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then

$$\dim_M(f(X)) \leq \dim_M(X)$$

where $f(X) \subset \mathbb{R}^m$ is the image of $X$ under $f$.

(c) Let $B \subset \mathbb{R}^n$ be a Besicovitch set, and consider $B \times B \subset \mathbb{R}^{2n}$. Let $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ be defined by

$$f(\vec{x}, \vec{y}) = \vec{x} - \vec{y}$$

where $\vec{x}, \vec{y} \in \mathbb{R}^n$, so that the ordered pair $(\vec{x}, \vec{y})$ is a point in $\mathbb{R}^{2n}$. Prove that $f(B \times B)$ contains a unit ball in $\mathbb{R}^n$. Since a unit ball in $\mathbb{R}^n$ has Minkowski dimension $n$, conclude that $\dim_M(B) \geq n/2$.

(d) By mimicking the above argument, prove the finite field analogue of this result; namely, prove that if $B \subset \mathbb{F}_q^n$ is a Besicovitch set, then

$$|B| \geq q^{n/2}$$

Up until Dvir proved the Finite Field Kakeya Conjecture, this was the sort of argument that was used all the time: taking a result in the analytic world and transferring it to the combinatorial world, or vice versa.

4* In this problem, you will be defining Hausdorff dimension. Note that this problem is both hard and not particularly relevant to what we're doing in class, so only do it if you're interested in what Hausdorff dimension is.

(a) The idea for Hausdorff dimension is as follows. For every real number $s \geq 0$, we will define a notion of $s$-dimensional volume; for $s \in \mathbb{N}$, this will agree (except for a caveat) with our usual notion of $s$-dimensional volume, namely length, area, etc. Convince yourself that if a set $S \subset \mathbb{R}^n$ "should have" dimension $d$, then we expect its $s$-dimensional volume to be $\infty$ when $s < d$ and $0$ when $s > d$. This is a generalization of the idea that a plane should have "length" $\infty$ but "volume" 0.

(b) This notion of $s$-dimensional measure is the so-called $s$-dimensional Hausdorff measure. Defining it is hard, so instead we'll define what's called the $s$-dimensional Hausdorff content of a set, which is actually a really bad notion of $s$-dimensional volume. For a set $S \subseteq \mathbb{R}^n$, its $s$-dimensional Hausdorff content is

$$C_H^s(S) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s \,\middle|\, \{U_i\}_{i=1}^{\infty} \text{ is a countable cover of } S \text{ by open sets} \right\}$$

where $\operatorname{diam}(U_i)$ denotes the diameter of $U_i$, namely the maximal distance between two points in $U_i$. If you don't know what inf means, just pretend it says min; they're more or less the same thing. Convince yourself that if $S$ is a line segment of length $L$, then $C_H^1(S) = L$, so that this is at least not an insane notion of 1-dimensional volume.

(c) On the other hand, prove that if $S$ is two line segments of length $L$ joined at a right angle (i.e. $S$ looks like $\llcorner$ ), then $C_H^1(S) < 2L$. This issue is resolved with the Hausdorff measure, but again, we're not taking that (better, but more complicated) approach.

(d) What is $C_H^0(S)$? Is this a reasonable notion of 0-dimensional volume?

(e)* Prove that if $r < s$ and $C_H^r(S) < \infty$, then $C_H^s(S) = 0$.

This (and also a similar fact that's a bit harder to prove) shows that if we plot a graph where the $x$-axis is the value of $s$ and the $y$-axis is the value of $C_H^s(S)$, then this graph will be $\infty$ for a while, then jump to 0 at some point (and we're not sure what it does at the jump point). This jump point, formally defined as $\inf\{s : C_H^s(S) = 0\}$, is called the *Hausdorff dimension* of $S$, and is denoted by $\dim_H S$.

(f) Prove that $\dim_H S \leq \dim_M S$. In particular, this implies that if $S \subseteq \mathbb{R}^n$ and $\dim_H S = n$, then $\dim_M S = n$. So the Kakeya Conjecture for Hausdorff dimension is stronger than that for Minkowski dimension.

(g) Prove that the Hausdorff dimension has the following nice property with regards to countable unions: if $\{S_i\}_{i=1}^{\infty}$ is a countable collection of sets, then

$$\dim_H \left( \bigcup_{i=1}^{\infty} S_i \right) = \sup\{\dim_H(S_i) : 1 \leq i < \infty\}$$

where again, if you don't know what sup means, pretend it says max.

(h) Using (4g), prove that $\dim_H \mathbb{Q} = 0$. On the other hand, prove that $\dim_M \mathbb{Q} = 1$. Thus, the inequality in part (4f) can be a strict inequality.

1 Prove, as I stated in class, that

$$\binom{q+n-1}{n} \geq \frac{1}{n!}q^n$$

This fact fully completes the proof of the Finite Field Kakeya Conjecture.

2 As I mentioned in class, Dvir-Kopparty-Saraf-Sudan improved Dvir's bound and proved that if $B \subseteq \mathbb{F}_q^n$ is a Besicovitch set, then $|B| \geq q^n/2^n$. The best-known upper bound, also due to Dvir, is *almost* equal to this lower bound: he gave a construction of a Besicovitch set $B_n$ with $|B_n| \approx q^n/2^{n-1}$, which is off by a factor of 2. In this exercise, you'll go through this construction.

(a) First, suppose that $q$ is odd, and $n \geq 1$. Let

$$D_n = \left\{(\alpha_1, \ldots, \alpha_{n-1}, \beta) \middle| \alpha_i, \beta_i \in \mathbb{F}_q, \alpha_i + \beta^2 \text{ is a square in } \mathbb{F}_q \text{ for all } i\right\}$$

Prove that

$$|D_n| = q\left(\frac{q+1}{2}\right)^{n-1}$$

*Hint:* There are exactly $(q+1)/2$ perfect squares in $\mathbb{F}_q$.

(b) Also, let $\mathbb{F}_q^{n-1} \times \{0\}$ denote the set $\{(\boldsymbol{a}, 0) \mid \boldsymbol{a} \in \mathbb{F}_q^{n-1}\}$. Finally, let

$$B_n = D_n \cup \left(\mathbb{F}_q^{n-1} \times \{0\}\right)$$

Conclude that

$$|B_n| \leq q\left(\frac{q+1}{2}\right)^{n-1} + q^{n-1} \approx \frac{q^n}{2^{n-1}}$$

(c) Finally, we need to check that $B_n$ is a Besicovitch set. For that, we need to check that for each $\boldsymbol{m} \in \mathbb{F}_q^n$, we have some $\boldsymbol{b} \in \mathbb{F}_q^n$ such that $\ell_{\boldsymbol{m},\boldsymbol{b}} \subseteq B$, where

$$\ell_{\boldsymbol{m},\boldsymbol{b}} = \{\boldsymbol{b} + t \cdot \boldsymbol{m} | t \in \mathbb{F}_q\}$$

First, suppose that $m_n$, the last coordinate of $\boldsymbol{m}$, is zero. Then prove that for $\boldsymbol{b} = 0$, we have that $\ell_{\boldsymbol{m},\boldsymbol{b}} \subseteq B_n$.

(d) On the other hand, suppose that $m_n \neq 0$. Then set

$$\boldsymbol{b} = \left(\left(\frac{m_1}{2m_n}\right)^2, \left(\frac{m_2}{2m_n}\right)^2, \ldots, \left(\frac{m_{n-1}}{2m_n}\right)^2, 0\right)$$

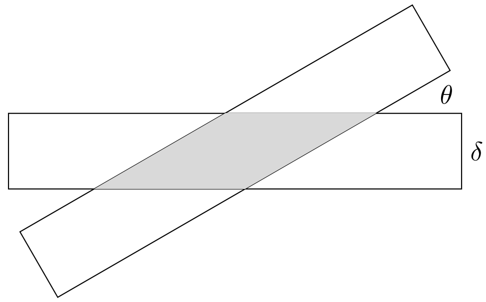
Note that this is well-defined since $q$ is odd. Then check that for each $t$, we have that $\boldsymbol{b} + t \cdot \boldsymbol{m} \in D_n$, so that $\ell_{\boldsymbol{m},\boldsymbol{b}} \subseteq B_n$. Therefore, $B_n$ is indeed a Besicovitch set.

(e)* If you're interested, also do the case where $q$ is even, which is a bit harder. In that case, set

$$B_n = \left\{(\alpha_i, \ldots, \alpha_{n-1}, \beta) \middle| \alpha_i, \beta \in \mathbb{F}_q, \exists \gamma_i \in \mathbb{F}_q \text{ such that } \alpha_i = \gamma_i^2 + \gamma_i \beta\right\}$$

Confirm that $|B_n| \approx q^n/2^{n-1}$ and that $B_n$ is a Besicovitch set.

3 Prove that if two $1 \times \delta$ rectangles $R_1, R_2$ make an angle $\theta$ between them, then the area of the intersection is at most $C\delta^2/\theta$, where $C$ is some universal constant:




This will be very useful tomorrow.

4* The way Dvir-Kopparty-Saraf-Sudan proved their stronger lower bound is by mimicking Dvir's polynomial method, but using *multiplicities;* specificially, they construct a polynomial that not only has a root at each point of $B$, but actually a multiple root at each point of $B$. Think about what sorts of theories you need to develop for this: a definition of multiple roots over finite fields, a Schwartz-Zippel analogue for roots with multiplicity, an upper bound lemma with multiplicities (if a set is small, then such a low-degree polynomial exists), and finally an actual proof of this stronger Kakeya theorem. See how many of these steps you can do by yourself, and if you want to see the full details, read this survey by Vsevolod Lev (this proof starts on page 3): http://math.haifa.ac.il/seva/Notes/Dvir_Kakeya.pdf