

1 Review

First, we're going to review some of the things that we did on the last day of the Kakeya Conjecture class. Recall that one of the many equivalent definitions of Minkowski dimension was the following:

Definition. Let $S \subset \mathbb{R}^n$ be a bounded subset, and let $\delta > 0$. By $N_\delta(S)$ we denote the δ -neighborhood of S , namely all points that are at distance $\leq \delta$ away from S . Formally,

$$N_\delta(S) = \{x \in \mathbb{R}^n : \exists y \in S \text{ such that } d(x, y) \leq \delta\}$$

where $d(x, y)$ is the Euclidean distance between x and y . Then, let

$$V(S, \delta) = \text{vol}(N_\delta(S))$$

Finally, the Minkowski dimension of S is given by

$$\dim_M S = \lim_{\delta \rightarrow 0} \left(n + \frac{\log V(S, \delta)}{\log \frac{1}{\delta}} \right)$$

In our proof of the Kakeya Conjecture in dimension 2, we used this definition. Specifically, we started with a Besicovitch set $B \subset \mathbb{R}^2$, and our goal was to show that $V(B, \delta)$ was reasonably large (in terms of δ). To do this, we picked a subset of $N_\delta(B)$ that consisted of a whole bunch of $1 \times \delta$ rectangles, and which pointed in a large number of “different” directions, and then we proved that the union of these rectangles must have fairly large area, namely at least $c/\log \frac{1}{\delta}$ for some constant c . Since this union of rectangles is a subset of $N_\delta(B)$, we concluded that $V(B, \delta)$ is fairly large, which sufficed to prove that $\dim_M B = 2$.

2 Tubes and Separation

We can generalize this idea to higher dimensions, as follows.

Definition. Let e be a direction in \mathbb{R}^n , which we can equivalently think of as a point $e \in S^{n-1}$, where S^{n-1} is the $(n-1)$ -dimensional sphere in \mathbb{R}^n . Then for any point $a \in \mathbb{R}^n$ and any $\delta > 0$, we will denote by $T_e^\delta(a)$ the tube of radius δ around a line segment of length 1 centered at a . Formally,

$$T_e^\delta(a) = \left\{ x \in \mathbb{R}^n : |(x-a) \cdot e| \leq \frac{1}{2}, \text{proj}_{e^\perp}(x-a) \leq \delta \right\}$$

where proj_{e^\perp} denotes the orthogonal projection onto the hyperplane perpendicular to e , and \cdot denotes the dot product of vectors in \mathbb{R}^n .

Definition. Given $\delta > 0$, a δ -separated set of directions in \mathbb{R}^n is a collection of directions $e_1, \dots, e_m \in S^{n-1}$ with the property that $\|e_i - e_j\| \geq \delta$ for all $1 \leq i \neq j \leq m$.

A *maximally* δ -separated set of directions is such a set that is maximal with respect to inclusion, namely that for every other point $e \in S^{n-1}$, there is some $1 \leq i \leq m$ such that $\|e - e_i\| < \delta$.

Proposition. For any n , there exist constants $c, C > 0$ such that if Ω is a maximally δ -separated set of directions in \mathbb{R}^n , then

$$c\delta^{1-n} \leq |\Omega| \leq C\delta^{1-n}$$

Proof idea. The surface area of S^{n-1} is some constant depending on n , and the surface area of a radius- δ disk on this sphere is some constant times δ^{n-1} . So just by dividing, we see that the size of some maximally δ -separated set Ω must be some constant multiplied by δ^{1-n} . \square

With these notions in hand, we can state an equivalent version of the Kakeya Conjecture for Minkowski dimension.

Theorem. *The Keakeya Conjecture for Minkowski dimension in \mathbb{R}^n is equivalent to the following statement: for any $\delta > 0, \varepsilon > 0$ and any maximally δ -separated set of directions e_1, \dots, e_m in \mathbb{R}^n , there is some constant c_ε depending only on ε such that*

$$\text{vol} \left(\bigcup_{i=1}^m T_{e_i}^\delta(a_i) \right) \geq c_\varepsilon \delta^\varepsilon$$

where a_1, \dots, a_m are any points in \mathbb{R}^n . In other words, no matter how we arrange $1 \times \delta$ tubes pointing in a maximally δ -separated set of directions, the volume of the union is pretty large, namely at least δ^ε (up to constants).

Proof. The idea is the same as what we did previously in dimension 2, though the notation is a bit more complicated. The important implication for us will be that if this statement about tubes is true, then the Keakeya Conjecture is true as well. For that, suppose that $B \subseteq \mathbb{R}^n$ is a Besicovitch set. We fix a maximally δ -separated set of directions e_1, \dots, e_m , and find $a_1, \dots, a_m \in B$ such that the line segment in direction e_i through a_i is a subset of B . Then we have that

$$\bigcup_{i=1}^m T_{e_i}^\delta(a_i) \subseteq N_\delta(B)$$

So we conclude that

$$V(B, \delta) \geq c_\varepsilon \delta^\varepsilon$$

for every $\varepsilon > 0$. Therefore, we get that

$$\lim_{\delta \rightarrow 0} \frac{\log V(B, \delta)}{\log \frac{1}{\delta}} \geq \lim_{\delta \rightarrow 0} \frac{\log c_\varepsilon + \varepsilon \log \delta}{-\log \delta} = -\varepsilon - \lim_{\delta \rightarrow 0} \frac{\log c_\varepsilon}{\log \delta} = -\varepsilon$$

Since this holds for any ε , we actually conclude that

$$\lim_{\delta \rightarrow 0} \frac{\log V(B, \delta)}{\log \frac{1}{\delta}} \geq 0$$

and thus

$$\dim_M B = \lim_{\delta \rightarrow 0} \left(n + \frac{\log V(B, \delta)}{\log \frac{1}{\delta}} \right) \geq n$$

which implies that the Keakeya Conjecture for Minkowski dimension is true in \mathbb{R}^n .

The reverse implication is more or less the same, though it's a bit more technical. We'll skip it, since our ultimate goal is to discuss a conjecture that's stronger than the Keakeya Conjecture. \square

Recall that there is another important notion of dimension that we didn't discuss in any detail, namely the Hausdorff dimension. We still won't define it, but let me give you one important result about it.

Theorem. *Fix $\delta > 0$, and let e_1, \dots, e_m be a maximally δ -separated set of directions in \mathbb{R}^n . Additionally, let $T_{e_1}^\delta(a_1), \dots, T_{e_m}^\delta(a_m)$ be any collection of $1 \times \delta$ tubes that point in the directions e_1, \dots, e_m . Let $\lambda = 1/\log^2 \frac{1}{\delta}$, and let A_1, \dots, A_m be any sets such that $A_i \subseteq T_{e_i}^\delta(a_i)$ and such that*

$$\text{vol}(A_i) \geq \lambda \text{vol}(T_{e_i}^\delta(a_i))$$

Then the Keakeya Conjecture for Hausdorff dimension is true in \mathbb{R}^n if and only if

$$\text{vol} \left(\bigcup_{i=1}^m A_i \right) \geq c_\varepsilon \lambda^n \delta^\varepsilon$$

for all $\varepsilon > 0$, where c_ε is some constant depending only on ε .

The idea here is that rather than getting a lower bound for the volume of a union of tubes that point in different directions, we're getting a bound for the volume of a union of reasonably large subsets of these tubes. We can get intuition for why λ^n is the correct correction factor as follows: suppose that all the a_i s are equal, namely all these tubes are centered at a single shared point. Additionally, suppose that the set A_i is simply the central $\lambda \times \delta$ subtube of the given $1 \times \delta$ tube. Then in the range of λ we are considering, these tubes have a maximal overlap, which means that $\bigcup_{i=1}^m A_i$ is basically a ball of radius $\lambda/2$. So

$$\text{vol} \left(\bigcup_{i=1}^m A_i \right) = C \lambda^n$$

and this λ^n is the reason λ^n is the right factor above.

When phrased in these ways, there is a generalization of both conjectures that is staring us in the face. This is (one of the many equivalent versions of) the Kakeya Maximal Conjecture:

Conjecture (Kakeya Maximal Conjecture). *Fix $\delta > 0$, and let e_1, \dots, e_m be a maximally δ -separated set of directions in \mathbb{R}^n . Additionally, let $T_{e_1}^\delta(a_1), \dots, T_{e_m}^\delta(a_m)$ be any collection of $1 \times \delta$ tubes that point in the directions e_1, \dots, e_m . Then for any $\lambda > 0$, and let A_1, \dots, A_m be any sets such that $A_i \subseteq T_{e_i}^\delta(a_i)$ and such that*

$$\text{vol}(A_i) \geq \lambda \text{vol}(T_{e_i}^\delta(a_i))$$

Then we conjecture that

$$\text{vol} \left(\bigcup_{i=1}^m A_i \right) \geq c_\varepsilon \lambda^n \delta^\varepsilon$$

for all $\varepsilon > 0$, where c_ε is some constant depending only on ε .

The crucial difference here is that we ask this to hold for any $\delta > 0$; the $\lambda = 1$ case is the Kakeya Conjecture for Minkowski dimension, and the $\lambda = 1/\log^2 \frac{1}{\delta}$ is the Kakeya Conjecture for Hausdorff dimension.

This is not how the Maximal Kakeya Conjecture is usually phrased. Usually, it is described as a “restricted weak-type L^p bound on the Kakeya Maximal Operator”; we’re not using this terminology because it requires a lot of integration theory and functional analysis as a prerequisite, and because I don’t actually understand it.

Why is the Kakeya Maximal Conjecture a useful conjecture to think about? There are a couple of reasons. First of all, the entire idea of thinking about volumes of tubes is, as we saw, very useful. In particular, this “discretized” problem allows people to bring in many geometric, algebraic, and arithmetic arguments that seem to not be available when one considers only amorphous “infinitary” Besicovitch sets. This is basically what we saw when we proved the Kakeya Conjecture in \mathbb{R}^2 , where we could use a simple property about how rectangles can intersect in the plane.

Additionally, the formulation of the conjecture in terms of the L^p bound on the Kakeya Maximal Operator, whatever that means, turns out to allow a whole set of other machinery to get involved. Namely, analysts have been spending years proving such bounds for other operators, and they’ve been able to use such techniques to get partial results about the Kakeya Maximal Conjecture.