0. Go through the examples in the lecture notes and make sure you understand them!

1. Let $p \in [0,1]$. Suppose we have a biased coin, which comes up heads with probability p and tails with probability $1-p$. If we flip the coin m times, what is the probability that it comes up heads exactly k of those m times, for each $0 \le k \le m$?

2. Let X be the outcome of a roll of a 20-sided die, with sides labeled $1, 2, \ldots, 20$. Let Y be the indicator random variable for the event that X is prime, and Z the indicator random variable for the even that X is even.

   (a) Compute the distributions of Y and Z (in other words, compute $\Pr(Y = 0)$ and $\Pr(Y = 1)$, and the same for Z).

   (b) Compute $\Pr(Y = 0 \mid Z = 1)$ and $\Pr(Z = 1 \mid Y = 1)$. If you'd like, compute some other things along these lines.

   (c) For each $1 \le k \le 20$, compute $\Pr(X = k \mid Y = 1)$ and $\Pr(X = k \mid Z = 0)$.
   (A formula for how this looks for general k is fine; you don't need to write down 40 numbers to answer this question.)

3. Let X, Y be random variables on state spaces $\mathcal{X}, \mathcal{Y}$, respectively (these are totally arbitrary random variables, which may be dependent or independent, and $\mathcal{X}, \mathcal{Y}$ might be any finite subsets of $\mathbb{R}$). Let $Z = X + Y$.

   (a) What is the state space $\mathcal{Z}$ of Z?

   (b) For each $z \in \mathcal{Z}$, find a formula for $\Pr(Z = z)$.

   (c) Let $W = XY$. Similarly, determine the state space of W and find a formula for $\Pr(W = w)$.

4. Let X be the outcome of a roll of a six-sided die. Let Y be the indicator random variable for the event that X is a perfect square, and Z the indicator random variable that X is even. Are Y and Z dependent or independent?

   Note: Independence basically means what you think it should mean, but sometimes surprises occur. I'm not sure that I would have expected this answer!

$\star$5. There is a subtlety I have glossed over in the definition of independence. It is possible to have a collection of three random variables X_1, X_2, X_3 such that any pair of them are independent, but if you know the value of any two of them, then this completely determines the value of the third.

   (a) Find an example of three such random variables.

   $\star\star$(b) Generalize this to more random variables: for every $m > k \ge 2$, find m random variables such that any k of them are independent, but knowing any k of them determines the values of the remaining $m - k$.

   Note: This distinction is between what is called *pairwise* (or *k-wise*) *independence* and *mutual independence*. Mutual independence means that each random variable is independent of any combination of the others; pairwise means that all pairs are independent.

   Second note: This distinction won't matter in this class, so you should feel free to not worry about it.

0. Let X, Y be random variables. Is the conditional expectation $\mathbb{E}[X \mid Y]$ a number, or a random variable?

1. Prove items (i)—(iii) in Theorem 2.9 in the lecture notes (reproduced here):

   (i) Conditional expectation is linear: $\mathbb{E}[X + Y \mid Z] = \mathbb{E}[X \mid Z] + \mathbb{E}[Y \mid Z]$ and $\mathbb{E}[\alpha X \mid Z] = \alpha\mathbb{E}[X \mid Z]$ for any real number α.

   (ii) If X is determined by Z, then $\mathbb{E}[X \mid Z] = X$.

   (iii) If X and Z are independent, then $\mathbb{E}[X \mid Z] = \mathbb{E}[X]$.

2. We enter the casino where the only game is the one where you bet on a coin toss, and win your bet if the coin comes up heads, and lose the bet if it comes up tails. Recall the "double your money" betting strategy from the blurb for this class, where on the nth turn you bet $\$2^n$, and leave the casino the first time it comes up heads.

   (a) In case you never read the blurb, convince yourself that this strategy wins you \$2 with probability 1.

   (b) Let X be the number of dollars you earn on the final coin toss, i.e. when it first comes up heads. What is $\mathbb{E}[X]$?

3. Suppose we keep rolling a fair die until we get a 6. Let Y be the number of rolls we do until we stop, and let X be the number of 1s that we see during this process.

   (a) For every $y \in \{0, 1, 2, \dots\}$, compute $\mathbb{E}[X \mid Y = y]$.

   (b) What is $\mathbb{E}[X \mid Y]$? You can (and should!) express this in terms of Y.

$\star$4. We proved in class that if X and Y are independent random variables, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Is the converse true? In other words, is it the case that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ implies that X and Y are independent?

5. Let π be a uniformly random permutation of $\{1, \dots, n\}$. (Uniformly random means that each permutation is equally likely, arising with probability $1/n!$.) Let X be the number of fixed points of π, i.e. the number of integers $x \in \{1, \dots, n\}$ so that $\pi(x) = x$. Compute $\mathbb{E}[X]$.

   Hint: Use linearity of expectation.

$\oplus$ means that this problem is not directly related to the content of the class, and is for general breadth and edification.

$\star$ means that this problem is harder than the other ones.

✧6. Suppose I have three dice, with the following numbers written on them:

A: $2, 2, 4, 4, 9, 9$

B: $1, 1, 6, 6, 8, 8$

C: $3, 3, 5, 5, 7, 7$

Let's also use A, B, C to denote the random variable that is the outcome of the rolls of the corresponding dice.

(a) Compute $\mathbb{E}[A], \mathbb{E}[B]$, and $\mathbb{E}[C]$.

(b) Compute $\Pr(A > B), \Pr(B > C)$, and $\Pr(C > A)$.

0. Read and understand the proof of Theorem 2.9(v).

1. Let $(X_i)_{i\geq 0}$ be a martingale with respect to $(Y_i)_{i\geq 0}$. Prove that for all $0 \leq m < n$,

$$\mathbb{E}[X_n \mid Y_0, \ldots, Y_m] = X_m.$$

In other words, a fair game is "fair at all levels". If I tell you what happened on the first m turns and then ask what your expected winnings are after $n - m$ other turns, the answer is just your winnings after time m.

2. Let $(X_i)_{i\geq 0}$ be a martingale with respect to $(Y_i)_{i\geq 0}$. Prove that for all $i \geq 1$,

$$\mathbb{E}[(X_i - X_{i-1}) \mid Y_0, \ldots, Y_{i-1}] = 0.$$

3. Let X be any random variable, and let $Y_0, Y_1, \ldots$ be a sequence of random variables. For $i \geq 0$, define $X_i = \mathbb{E}[X \mid Y_0, \ldots, Y_i]$.

Prove that $X_0, X_1, X_2, \ldots$ is a martingale with respect to $Y_0, Y_1, Y_2, \ldots$.

Note: This is called the *Doob martingale*, and its extremely general definiton (we assumed *nothing*) helps explain why martingales are so useful and prevalent. One way of thinking about this is that learning the values of $Y_0, Y_1, \ldots$ reveals some information to us about the value of X; because of this, the Doob martingale is sometimes called the *information exposure martingale*. This says that the expected change in the ith step of the process, given the history, is zero—i.e. that this is a fair game.

4. Let $(X_i)_{i\geq 0}$ and $(X'_i)_{i\geq 0}$ be two sequences of random variables, each of which is a martingale with respect to $(Y_i)_{i\geq 0}$.

   (a) Is the sum sequence $(X_i + X'_i)_{i\geq 0}$ a martingale with respect to $(Y_i)_{i\geq 0}$?

   (b) Is the product sequence $(X_i X'_i)_{i\geq 0}$ a martingale with respect to $(Y_i)_{i\geq 0}$?

$\star$5. Suppose we shuffle a deck of cards, then lay them all out on the table face down. Let $Y_0 = 0$, and for $1 \leq i \leq 52$, let

$$Y_i = \begin{cases} 1 & \text{if the } i\text{th card is red,} \\ 0 & \text{if the } i\text{th card is black.} \end{cases}$$

Now we begin flipping the cards over from left to right, i.e. we reveal the value of Y_1, then the value of Y_2, and so on. For $i \geq 1$, let X_i be the proportion of face-down cards that are red after we've flipped over the first $i - 1$ cards.

   (a) Prove that

$$X_i = \frac{26 - Y_1 - \cdots - Y_{i-1}}{53 - i}.$$

   (b) Prove that $(X_i)_{1\leq i\leq 52}$ is a martingale with respect to $(Y_i)_{0\leq i\leq 52}$.

$\star$6. Re-solve Problem 5(b) using Problem 3.

7. For any of the martingales mentioned above, in class, or in the notes, come up with a prophecy-free sequence and check that the martingale transform is another martingale.

8. Let S, T be stopping times with respect to some sequence $(Y_i)_{i\geq 0}$. Prove that the following are all stopping times as well.

   (a) $\max(S, T)$

   (b) $\min(S, T)$

   (c) $S + T$

Problems from yesterday's homework you couldn't solve

The following problems from yesterday's homework were impossible because we didn't get to the definition of stopping times (for one of them) and because I didn't tell you about the general tower property.

1. Let $(X_i)_{i\geq 0}$ be a martingale with respect to $(Y_i)_{i\geq 0}$. Prove that for all $0 \leq m < n$,

   $$\mathbb{E}[X_n \mid Y_0, \ldots, Y_m] = X_m.$$

   In other words, a fair game is "fair at all levels". If I tell you what happened on the first m turns and then ask what your expected winnings are after $n - m$ other turns, the answer is just your winnings after time m.

2. Let X be any random variable, and let $Y_0, Y_1, \ldots$ be a sequence of random variables. For $i \geq 0$, define $X_i = \mathbb{E}[X \mid Y_0, \ldots, Y_i]$.

   Prove that $X_0, X_1, X_2, \ldots$ is a martingale with respect to $Y_0, Y_1, Y_2, \ldots$.

   Note: This is called the *Doob martingale*, and its extremely general definiton (we assumed *nothing*) helps explain why martingales are so useful and prevalent. One way of thinking about this is that learning the values of $Y_0, Y_1, \ldots$ reveals some information to us about the value of X; because of this, the Doob martingale is sometimes called the *information exposure martingale*. This says that the expected change in the ith step of the process, given the history, is zero—i.e. that this is a fair game.

3. Let S, T be stopping times with respect to some sequence $(Y_i)_{i\geq 0}$. Prove that the following are all stopping times as well.

   (a) $\max(S, T)$

   (b) $\min(S, T)$

   (c) $S + T$

New problems

4. Read and understand Example 5.5 in the notes, and make sure you understand the (subtle but important) distinction between the stopped process $(X_i^T)_{i\geq 0}$ and the random variable X_T.

5. Let $(Y_i)_{i\geq 0}$ be a sequence of random variables, and let $(X_i)_{i\geq 0}$ be another sequence with the property that X_i is determined by $Y_0, \ldots, Y_i$ for all i. Prove that $(X_i)_{i\geq 0}$ is a martingale with respect to $(Y_i)_{i\geq 0}$ if and only if

   $$\mathbb{E}[X_i - X_{i-1} \mid Y_0, \ldots, Y_{i-1}] = 0.$$

   (You proved one direction on yesterday's homework.)

6. We technically only defined the martingale transform sequence $((W \bullet X)_i)_{i\geq 0}$ when $(W_i)_{i\geq 1}$ is a prophecy-free sequence and $(X_i)_{i\geq 0}$ is a martingale, but the definition makes sense for any two sequences of random variables. Find examples showing that the transformed sequence $((W \bullet X)_i)_{i\geq 0}$ is *not* a martingale if $(W_i)_{i\geq 1}$ is not prophecy-free, or if $(X_i)_{i\geq 0}$ is not a martingale.

$\star$7. Let $(X_i)_{i\geq 0}$ be a martingale with respect to $(Y_i)_{i\geq 0}$. For $i \geq 0$, let

$$V_i = X_i^2 - \mathbb{E}\left[(X_1 - X_0)^2 + (X_2 - X_1)^2 + \cdots + (X_i - X_{i-1})^2 \mid Y_0, \ldots, Y_{i-1}\right].$$

(a) Prove that $(V_i)_{i\geq 0}$ is a martingale with respect to $(Y_i)_{i\geq 0}$.

(b) Conclude that

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_0^2] + \sum_{k=1}^{i} \mathbb{E}\left[(X_k - X_{k-1})^2\right].$$