Matrix spaces and graphs

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With Yinan Li, Youming Qiao, Avi Wigderson, & Chuanqi Zhang

Outline

- Matrix spaces and why we care about them
- Graphs and matrix spaces of restricted support
- Properties of matrix spaces and properties of graphs
  - Singularity
  - Nilpotency
  - Isomorphism
- **Inhertied correspondences:** Deep connections between linear algebra and graph theory

Warmup

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Theorem (Kabanets–Impagliazzo 2004)

*An efficient deterministic algorithm implies that **"VP $\neq$ VNP"**.*

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Example: $\boldsymbol{P} =$ singularity. Determining whether every every $M \in \boldsymbol{S}$ is singular is the problem from the previous slide.

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bipartite or directed graph H $\rightsquigarrow$ matrix space $\boldsymbol{S}_H$

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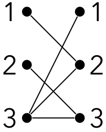

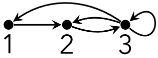

$\rightsquigarrow$

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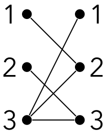

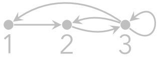

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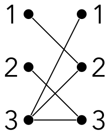

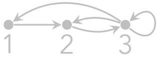

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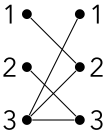

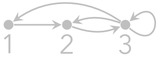

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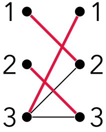

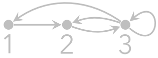

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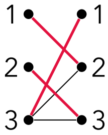

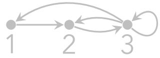

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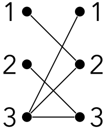

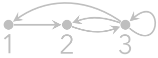

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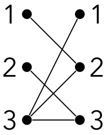

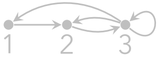

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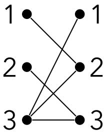

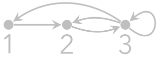

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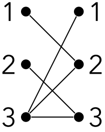

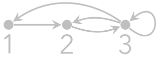

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This yields a randomized algorithm for bipartite perfect matching.

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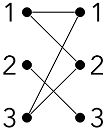

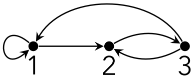

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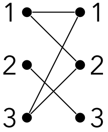

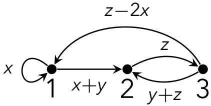

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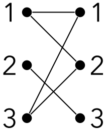

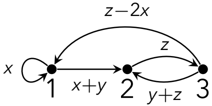

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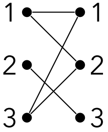

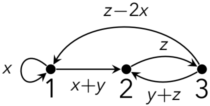

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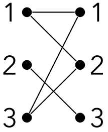

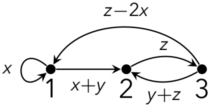

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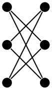
$H =$  $\iff$ $\boldsymbol{S}_H = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$

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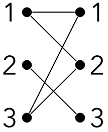

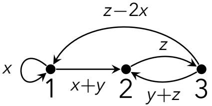

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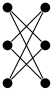
$H =$ 

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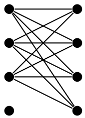
Fact

If a bipartite graph G on $n + n$ vertices has no perfect matching, then $e(G) \leq n^2 - n$.



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A combinatorial "explanation" of an algebraic property!

Basic and inherited correspondences

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Given such $\boldsymbol{S} \subseteq \boldsymbol{S}_G$, we efficiently and deterministically construct such $H \subseteq G$ with $e(H) = \dim \boldsymbol{S}$.

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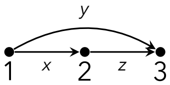

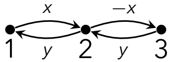

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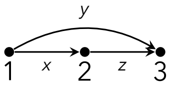

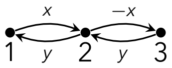

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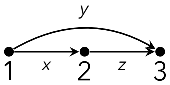

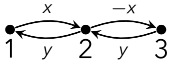

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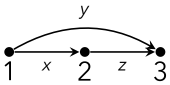

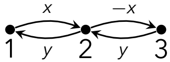

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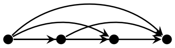

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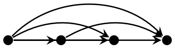

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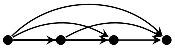

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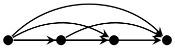

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We adapt de Seguins Pazzis's proof of Gerstenhaber's theorem.

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Fact: There is **no** inherited correspondence extending (a) $\Longleftrightarrow$ (b).

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$$\begin{array}{c}\text{Max. dim}\,\boldsymbol{S}\\ \text{for } \boldsymbol{S} \subseteq \boldsymbol{S}_G \text{ satisfying } \boldsymbol{P}\end{array} = \begin{array}{c}\text{Max. } e(H)\\ \text{for } H \subseteq G \text{ satisfying } \boldsymbol{Q}\end{array}$$

- Graphs help us understand matrix spaces (e.g. generalizations of Dieudonné and Gerstenhaber's theorems).
- For certain properties, matrix spaces are surprisingly rigid:
  - The lattice of subspaces of $\boldsymbol{S}_G$ is "not much richer" than the lattice of subgraphs of G.
  - The action of $\mathrm{GL}_n(\mathbb{F})$ is "not much richer" than that of S_n.

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This generalizes both Dieudonné and Gerstenhaber's theorems.

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  - A characterization may give a unified proof of Dieudonné and Gerstenhaber's theorems.
  - A characterization may resolve the conjecture on the previous slide, generalizing Atkinson's theorem.

Thank you!