Matrix spaces and graphs

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With Yinan Li, Youming Qiao, Avi Wigderson, & Chuanqi Zhang

Outline

- Matrix spaces and why we care about them
- Graphs and matrix spaces of restricted support
- Properties of matrix spaces and properties of graphs
  - Singularity
  - Nilpotency
  - Isomorphism
- **Inhertied correspondences:** Deep connections between linear algebra and graph theory

Warmup

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Theorem (Kabanets–Impagliazzo 2004)

*An efficient deterministic algorithm implies that **"P $\neq$ NP"**.*

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Example: $\boldsymbol{P}$ = singularity. Determining whether every every $M \in \boldsymbol{S}$ is singular is the problem from the previous slide.

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bipartite or directed graph H $\rightsquigarrow$ matrix space $\boldsymbol{S}_H$

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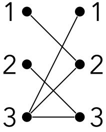

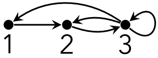

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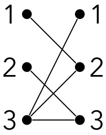

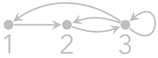

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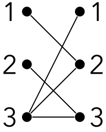

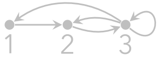

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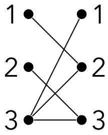

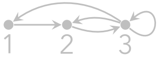

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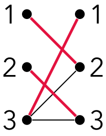

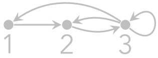

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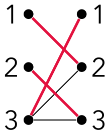

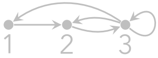

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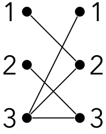

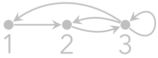

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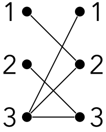

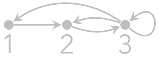

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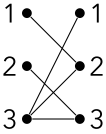

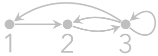

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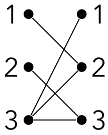

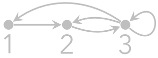

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This yields a randomized algorithm for bipartite perfect matching.

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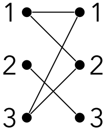

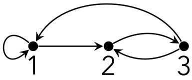

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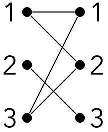

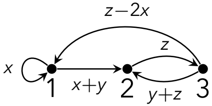

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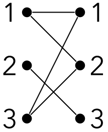

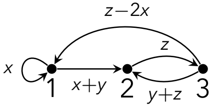

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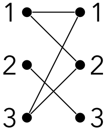

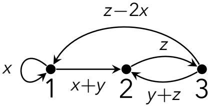

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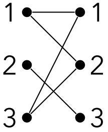

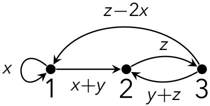

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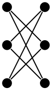
$H =$  $\iff$ $\boldsymbol{S}_H = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$

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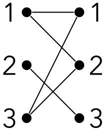

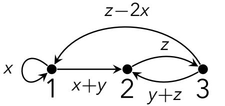

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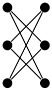
$H =$  $\iff$ $\boldsymbol{S}_H = \begin{pmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{pmatrix}$ $\quad \boldsymbol{S} = \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{pmatrix} \subseteq \boldsymbol{S}_H$

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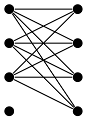
Fact

If a bipartite graph G on $n + n$ vertices has no perfect matching, then $e(G) \leq n^2 - n$.



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A combinatorial "explanation" of an algebraic property!

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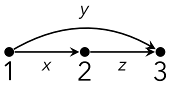

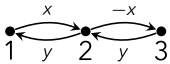

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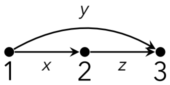

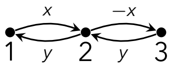

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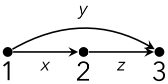

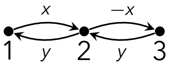

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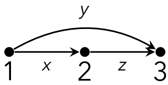

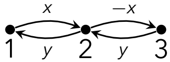

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Corollary

$\boldsymbol{S}_H$ is nilpotent $\iff$ *H is acyclic*

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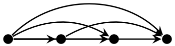

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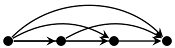

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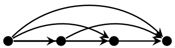

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Corollary: Given $\boldsymbol{T} \subseteq \mathbb{F}^{n\times n}$, it is NP-hard to determine the max. dimension of nilpotent $\boldsymbol{S} \subseteq \boldsymbol{T}$.

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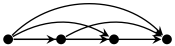

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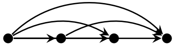

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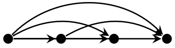

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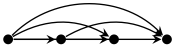

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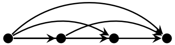

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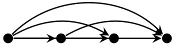

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Using this lemma and induction, given nilpotent $\boldsymbol{S} \subseteq \boldsymbol{S}_G$, we can efficiently and deterministically find acyclic $H \subseteq G$ with $e(H) = \dim \boldsymbol{S}$.

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Fact: There is **no** inherited correspondence extending (a) $\Longleftrightarrow$ (b).

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- Generalizes the action of $S_n \times S_n$, permuting the vertices of bipartite graphs.

The group $\mathrm{GL}_n(\mathbb{F})$ acts on $\mathbb{F}^{n \times n}$ by the **conjugation action**.

- Corresponds to a single change of basis in $\mathbb{F}^n$.
- This action is natural when we identify $\mathbb{F}^{n \times n}$ with $\mathrm{End}(\mathbb{F}^n)$, and when we care about multiplying matrices.
- Certain additional properties are invariant, e.g. nilpotency.
- Generalizes the action of S_n, permuting the vertices of directed graphs.

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- Graphs help us understand matrix spaces (e.g. generalizations of Dieudonné and Gerstenhaber's theorems).
- For certain properties, matrix spaces are surprisingly rigid:
  - The lattice of subspaces of $\boldsymbol{S}_G$ is "not much richer" than the lattice of subgraphs of G.
  - The action of $\mathrm{GL}_n(\mathbb{F})$ is "not much richer" than that of S_n.

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- Is there some general characterization (or necessary/sufficient conditions) of which properties have inherited correspondences?
  - Why is the structure of $\boldsymbol{S}_G$ not much richer than that of G?
  - A characterization may give a unified proof of Dieudonné and Gerstenhaber's theorems.

Thank you!

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This generalizes both Dieudonné and Gerstenhaber's theorems.