The Ramsey-Theoretic Monster Mash

Boo-val Wigderson

ITS Fellows' Seminar

October 31, 2023

Dreadful was the din
Of hissing through the hall, thick swarming now
With complicated monsters head and tail

John Milton, *Paradise Lost* X.521–3

Outline

Introduction: behemoths of Ramsey theory

Ghosts of graph Ramsey theory

Sea monsters and Ramsey multiplicity

Shapeshifters and oriented Ramsey numbers

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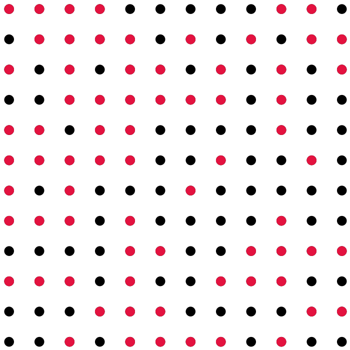

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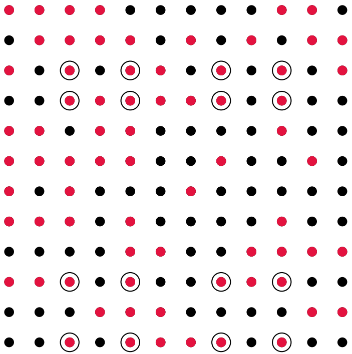

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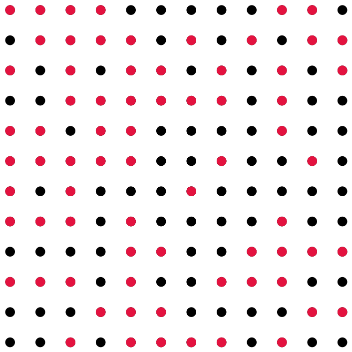

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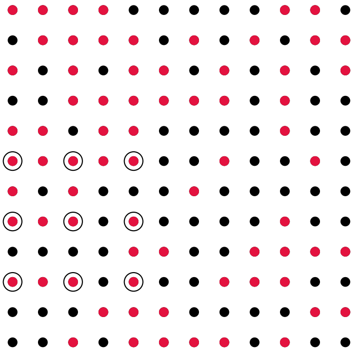

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Such results exist for integers, graphs, posets, Banach spaces…

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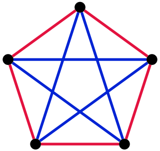

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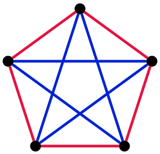

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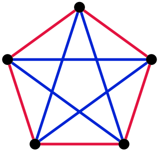

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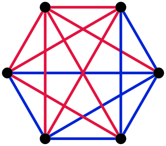

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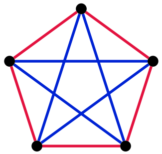

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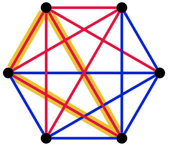

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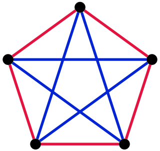

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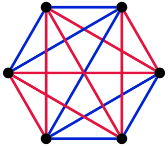

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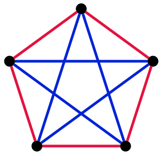

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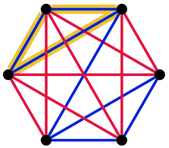

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So there exists a coloring of $E(K_N)$ with < 1 monochromatic K_t. □

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Theorem (Li 2023)

There exists an explicit coloring on $N \geq 2^{t^{0.00001}}$ vertices with no monochromatic K_t.

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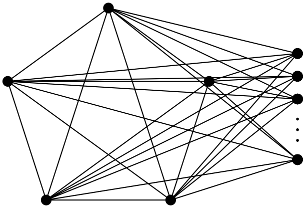

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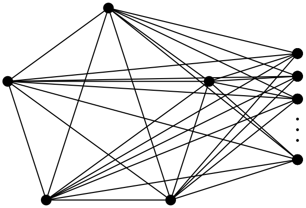

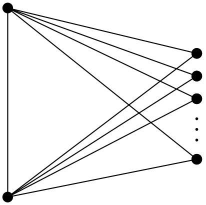

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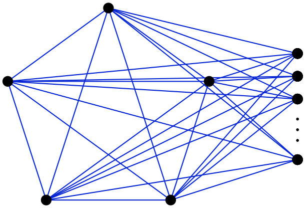

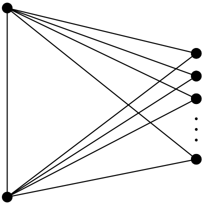

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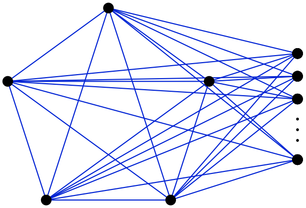

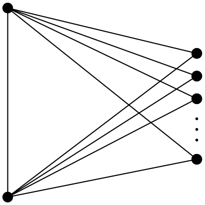

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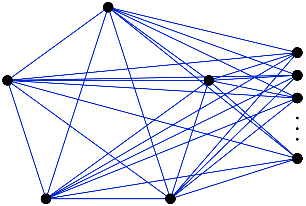

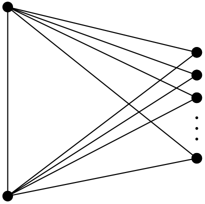

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In the n "page" vertices, it suffices to find a red K_t or a blue K_{t-k}.

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This result is still far too weak to improve the bound $r(t) < 4^t$.

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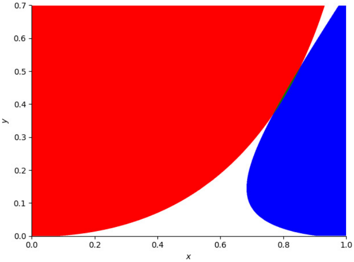

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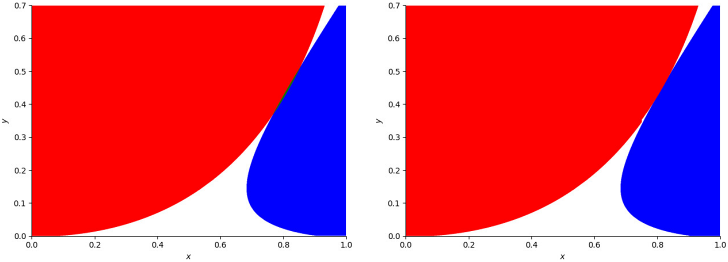
They introduced a “book algorithm” which can find some appropriate monochromatic book.



Outline

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Ghosts of graph Ramsey theory

Sea monsters and Ramsey multiplicity

Shapeshifters and oriented Ramsey numbers

Edges vs. triangles

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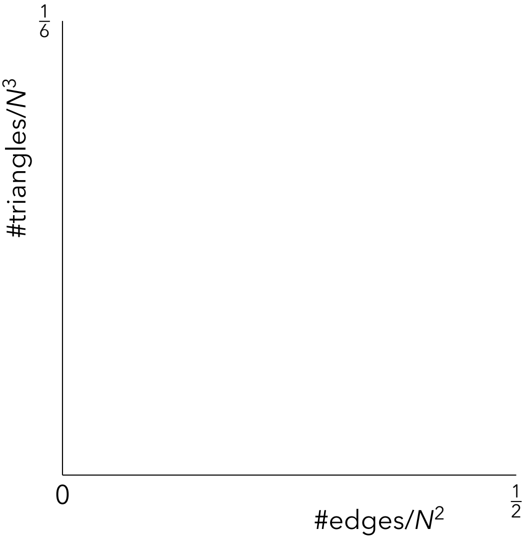

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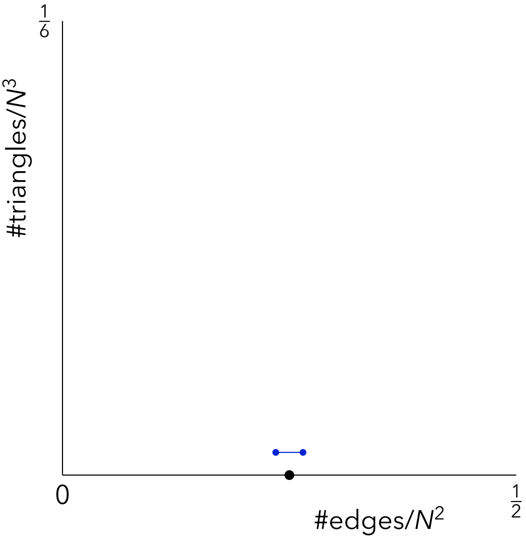

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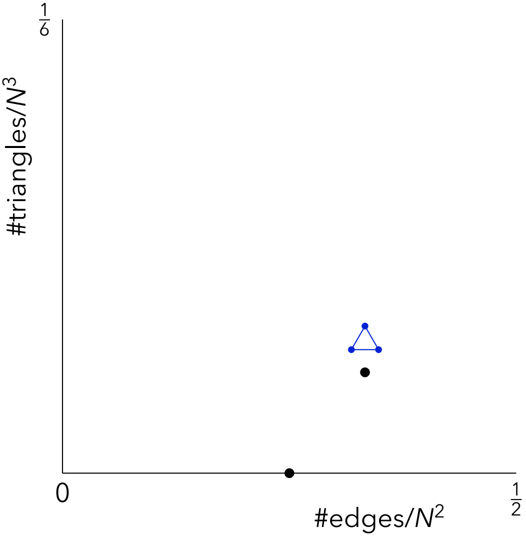

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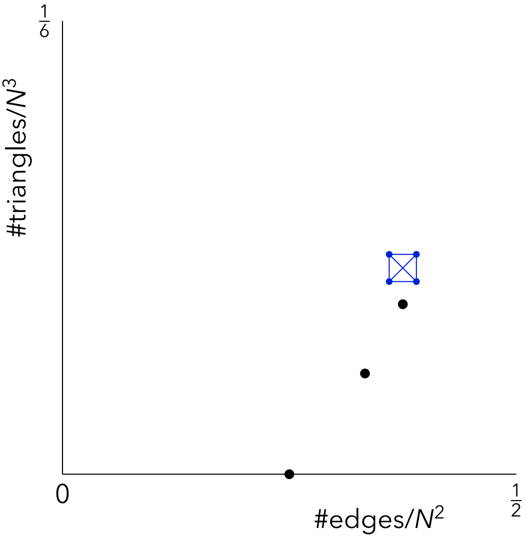

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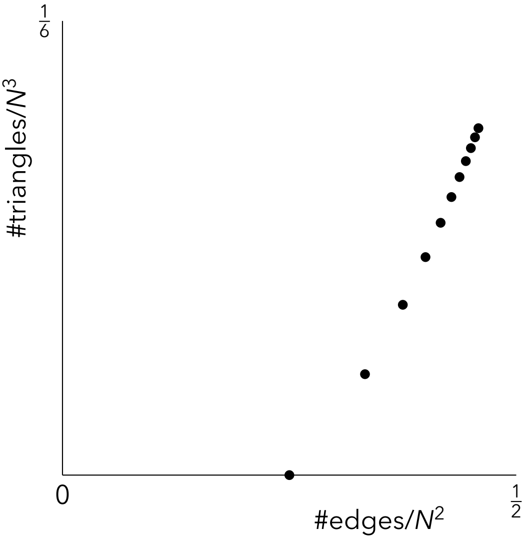

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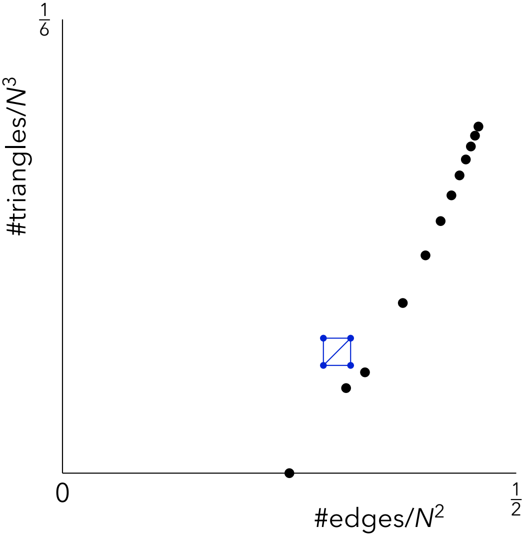

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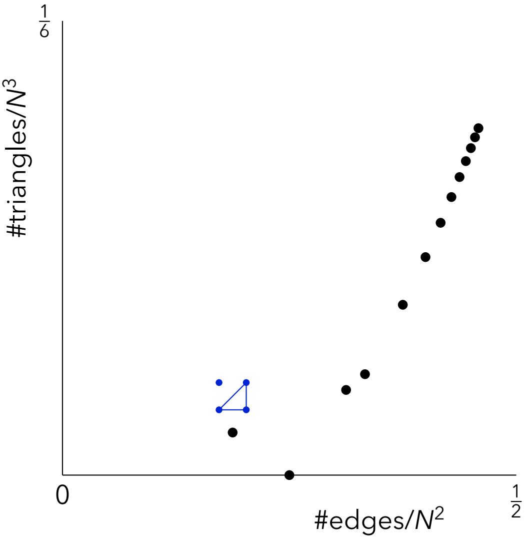

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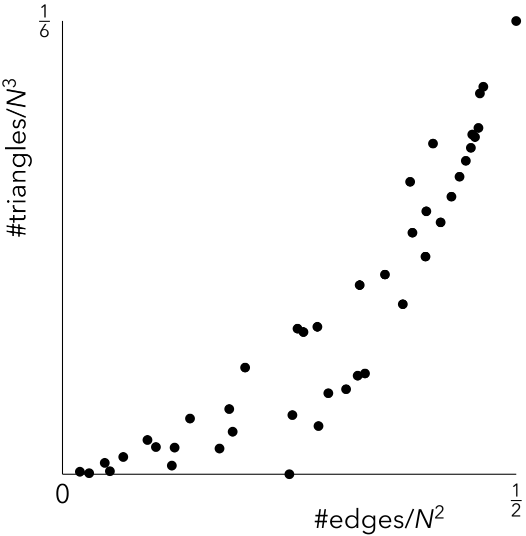

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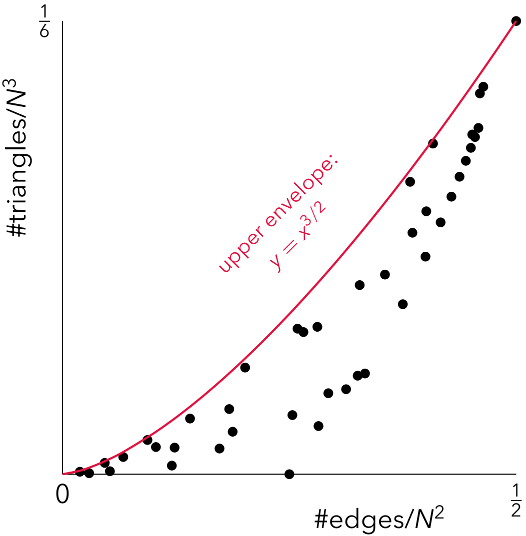

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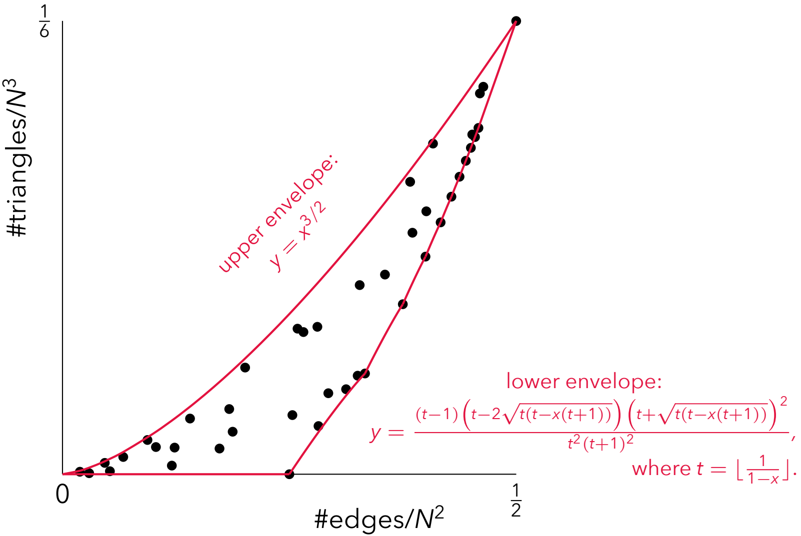

Densities

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The lower envelope for edges vs. K_3 is given by the function

$$y = \frac{(t-1)\ \left(t - 2\sqrt{t(t-x(t+1))}\right)\left(t + \sqrt{t(t-x(t+1))}\right)^{2}}{(t(t+1))^{2}},$$

where $t = \lfloor \frac{1}{1-x} \rfloor$.

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m-fold cover of an orthogonal tower with maximal Witt index.

A simpler monster

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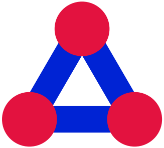
If H has chromatic number k and $\gg k^2$ edges, the Turán coloring beats the random coloring.



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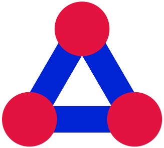

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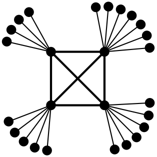

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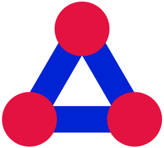

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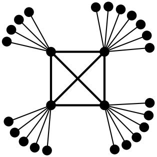

Open problem: Which coloring minimizes the number of K_4?

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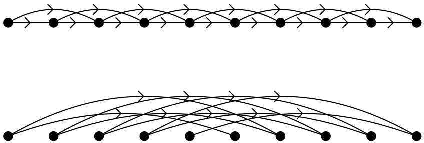

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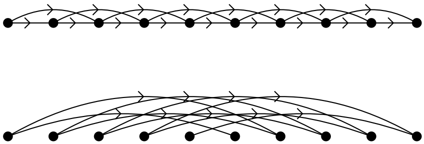

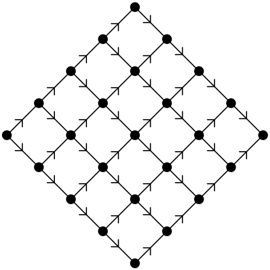

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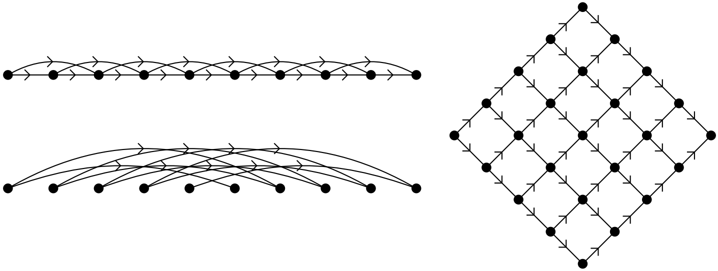

The digraphs for which $\vec{r}(H)$ is "large" are shapeshifters: they have many edges at every length scale, despite having bounded degree.

Thank you!