Lower bounds for multicolor Ramsey numbers

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SIAM Conference on Discrete Mathematics

June 15, 2022

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So there exists a coloring of $E(K_N)$ with < 1 monochromatic K_t. □

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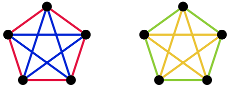

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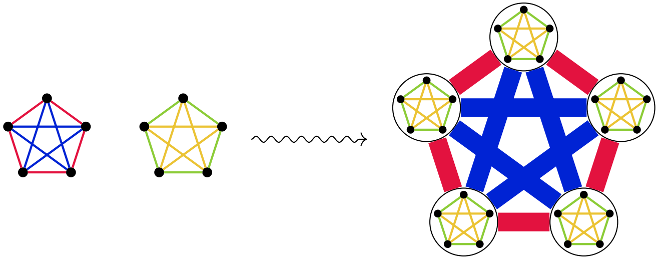

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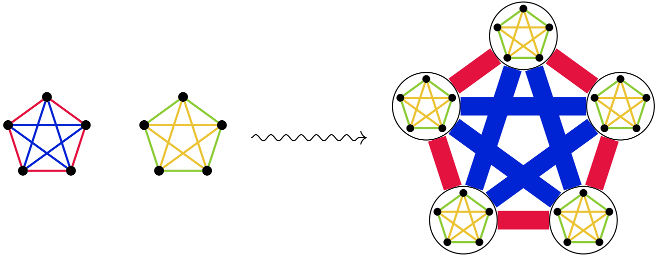

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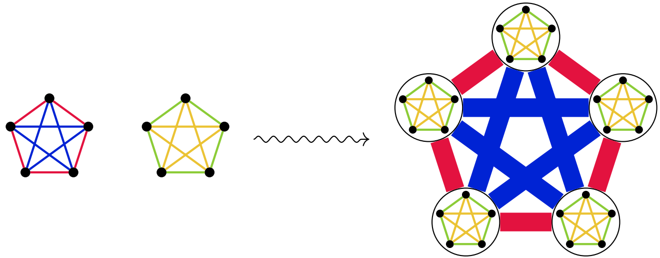

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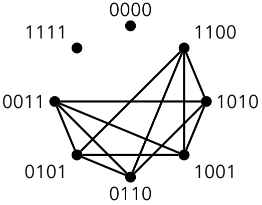

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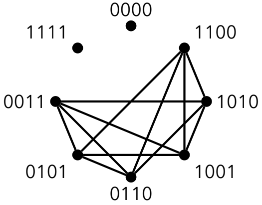

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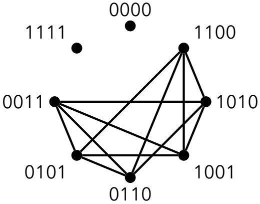

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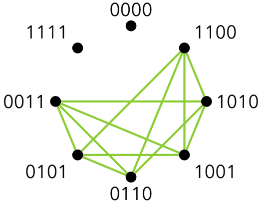

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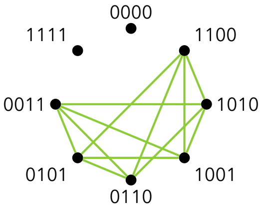

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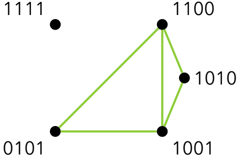

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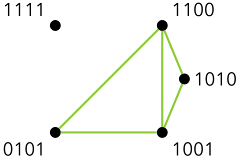

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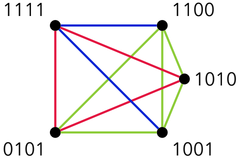

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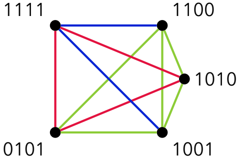

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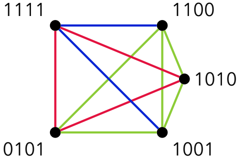

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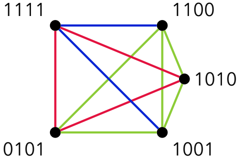

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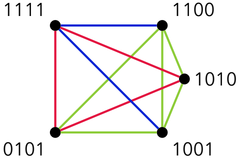

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No green K_t by Fact 1, so $r(t;3) > N \approx p|V_t| = 2^{\frac{7}{8}t - o(t)}$.

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For $q > 4$, Conlon and Ferber use the product coloring.

A new approach for more colors

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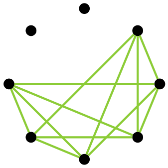
Overlay two random copies of G_t in green and yellow.



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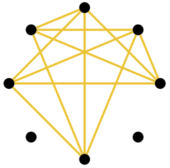

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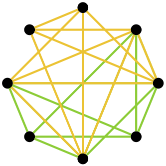

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Overlay two random copies of G_t in green and yellow.
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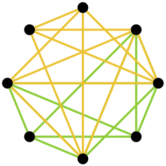

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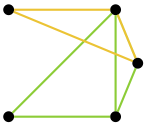

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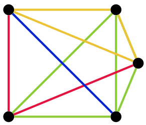

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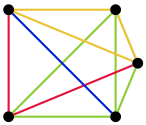

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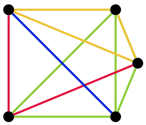

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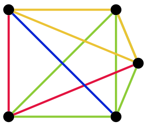

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How are we picking $p > 1$???

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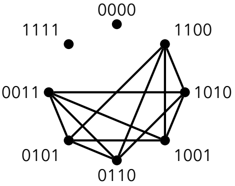

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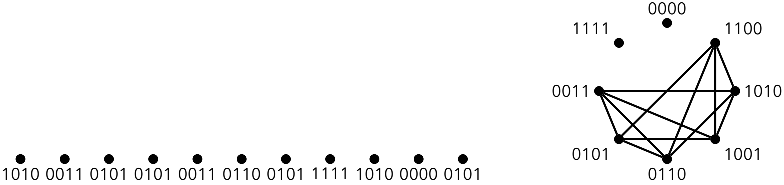

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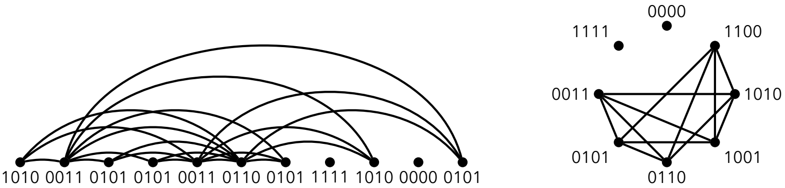

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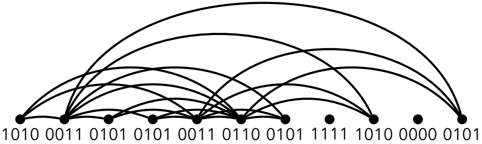

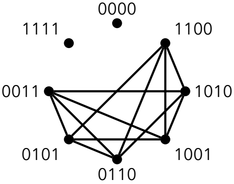

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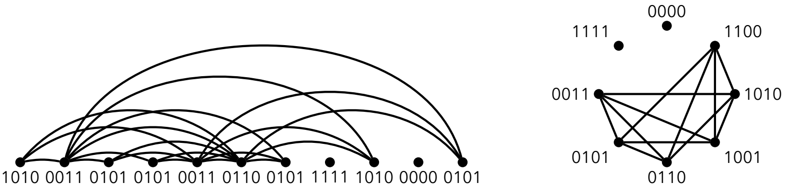

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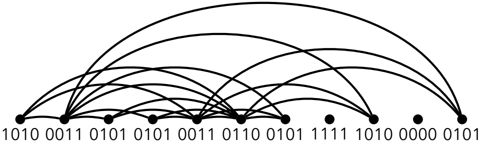

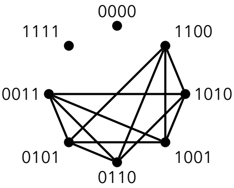

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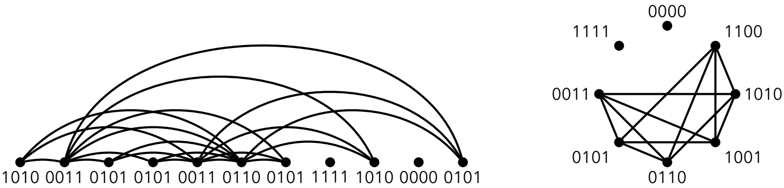

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So the above argument works for any p, if interpreted correctly.

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A better choice is $G(n,p)$ with $p = 0.454997$ and $n = p^{-t/2}$. □

Thank you!