1. (a) Prove that for every $k, q \geqslant 2$, the following holds. If $N \geqslant 4^k$, then in any q-coloring of $E(K_N)$ there is some copy of K_k whose edges receive at most $\lceil q/2 \rceil$ colors.

   (b) Prove that for any even integer $q \geqslant 2$ and any sufficiently large k, there exists a q-coloring of $E(K_N)$, where $N = 2^{k/2}$, such that every K_k receives more than $q/2$ colors.

Solution.

(a) Fix a q-coloring χ of $E(K_N)$, where $N \geqslant 4^k$. Arbitrarily divide the set of colors into two sets C_1, C_2, where $|C_1| = \lceil q/2 \rceil$ and $|C_2| = \lfloor q/2 \rfloor$. We may define a new coloring $\psi : E(K_N) \to \{1, 2\}$ by setting $\psi(e) = 1$ if $\chi(e) \in C_1$, and $\psi(e) = 2$ if $\chi(e) \in C_2$, for all $e \in E(K_N)$. By Theorem 2.1.4, we have that $N \geqslant 4^k \geqslant r(k)$, so there exists a monochromatic K_k under ψ. But this is precisely a set of k vertices all of whose edges receive a color from the same set C_i; in particular, since $|C_2| \leqslant |C_1| \leqslant \lceil q/2 \rceil$, this is a K_k whose edges receive at most $\lceil q/2 \rceil$ colors.

(b) Fix an even integer $q \geqslant 2$ and let k be sufficiently large with respect to q. Consider a random q-coloring of $E(K_N)$, where $N = 2^{k/2}$. For any given set of k vertices, the probability that the edges they define receive at most $q/2$ colors is at most $\binom{q}{q/2} 2^{-\binom{k}{2}}$. Indeed, we have $\binom{q}{q/2}$ choices for the $q/2$ colors to use, and having fixed these colors, each edge receives one of them with probability exactly $\frac{1}{2}$. Therefore, by the union bound, the probability that some k-set receives at most $q/2$ colors is at most

$$\binom{N}{k}\binom{q}{q/2} 2^{-\binom{k}{2}} \leqslant \frac{2^q 2^{k/2}}{k!} \cdot N^k 2^{-k^2/2} = \frac{2^q 2^{k/2}}{k!} \cdot \left(N 2^{-k/2}\right)^k = \frac{2^q 2^{k/2}}{k!}.$$

If k is sufficiently large in terms of q (which we assumed), this quantity is less than 1, since $2^{k/2}/k! \to 0$ as $k \to \infty$. Therefore, with positive probability, this coloring contains no K_k whose edges receive at most $q/2$ colors. Thus, there exists a coloring with this property, as desired. □

2. Prove that for every $k \geqslant 3$, there exists some N so that the following holds. Among any N points in the plane, there are either k points lying on a line, or k points in convex position.

Solution. Let $N = r_3(k, \mathrm{Kl}(k))$, and fix N points $p_1, \ldots, p_N$ in the plane. Define a 2-coloring of $E(K_N^{(3)})$ as follows: if a triple $p_i p_j p_\ell$ is collinear, we color $\{i, j, k\}$ red, and otherwise we color this triple blue. By the choice of N, there is either a red $K_k^{(3)}$ in this coloring, or a blue $K_{\mathrm{Kl}(k)}^{(3)}$. In the former case, we have found k points, such that every three of them are collinear, which is precisely a set of k collinear points. In the latter case, where we have a blue $K_{\mathrm{Kl}(k)}^{(3)}$, we have found $\mathrm{Kl}(k)$ points such that no three of them are collinear. But then, by the definition of $\mathrm{Kl}(k)$ (that is, by Theorem 10.3.4), these $\mathrm{Kl}(k)$ points contain k points in convex position. $\square$

3. Prove that
$$\lim_{k\to\infty} \frac{r(3,3,k)}{r(3,k)} = \infty.$$

Solution. By Theorem 2.1.4, we know that $r(3,k) \leqslant k^2$. We now claim that $r(3,3,k) \geqslant ck^3/(\log k)^6$ for an absolute constant $c > 0$; note that this implies the claimed result, since
$$\lim_{k\to\infty} \frac{r(3,3,k)}{r(3,k)} \geqslant \lim_{k\to\infty} \frac{ck^3/(\log k)^6}{k^2} = \lim_{k\to\infty} \frac{ck}{(\log k)^6} = \infty.$$
So it suffices to prove the claim. To do so, fix an integer k. By Bertrand's postulate, there exists a prime power q satisfying $k/(60(\ln k)^2) \leqslant q \leqslant k/(30(\ln k)^2)$, which implies $k \geqslant 30q(\ln q)^2$. By Lemma 4.3.7, there exists a triangle-free N-vertex graph G_q, where $N = q^3$, with at most M^k independent sets of order at most k, where $M = 200q/\ln q$.

We now apply Lemma 3.1.2 to conclude that
$$r(3,3,k) \geqslant \frac{N^2}{2ekM^2} = \frac{q^6}{2ek(200q/\ln q)^2} \geqslant 10^{-5}\frac{q^4(\ln q)^2}{k} \geqslant c\frac{k^3}{(\log k)^6},$$
for some absolute constant $c > 0$, where the last step follows from our lower bound on q. This is exactly what we wanted to prove. □

4. Let $k \geqslant 2$ be an integer, let $N = 16^k$, and let G be an N-vertex graph with at least $N^2/4$ edges.

   (a) Prove that there is a subset $T \subseteq V(G)$ with $|T| \geqslant \sqrt{N} - 1$ such that every set of k vertices in T has at least $\sqrt{N}$ common neighbors in G.

   (b) Prove that there are disjoint sets $A, B \subseteq V(G)$ with $|A| = |B| = k$ such that the following properties hold:

   - A is a clique or an independent set in G,
   - B is a clique or an independent set in G, and
   - all pairs $(a, b) \in A \times B$ are edges of G.

Solution.

(a) Note that since G has at least $N^2/4$ edges, it has average degree $d \geqslant N/2$. Let $\Delta = k, r = \sqrt{N} = 4^k, s = \sqrt{N} - 1$, and $t = 2k$. Notice that

$$\frac{d^t}{N^{t-1}} - \binom{N}{\Delta}\left(\frac{r}{N}\right)^t \geqslant N\left(\frac{d}{N}\right)^t - N^k\left(\frac{1}{\sqrt{N}}\right)^t \geqslant N2^{-t} - 1 = \sqrt{N} - 1 = s.$$

Therefore, by Lemma 5.4.11, there exists a set $T \subseteq V(G)$ with $|T| \geqslant s$ such that every Δ-element subset of T has at least r common neighbors in G; this is precisely what we wanted to show.

(b) Let T be the set given by part (a). Recall that $|T| \geqslant \sqrt{N} = 4^k$. By Theorem 2.1.4, we have that $r(k) < \sqrt{N}$, thus there exists a set $A \subseteq T$ such that A with $|A| = k$ such that A is either a clique or an independent set in G. By the definition of T, there is a set $W \subseteq V(G)$ with $|W| \geqslant \sqrt{N} - 1$ such that all vertices in A are adjacent to all vertices in W. Again applying the fact that $r(k) \leqslant 4^k - 1 \leqslant |W|$, we find a set $B \subseteq W$ with $|B| = k$ which is a clique or an independent set in G. Since all vertices of A are adjacent to all vertices of $W \supseteq B$, this is the desired configuration. $\square$

5. Prove that for every $q \geqslant 2$, there exists some N such that the following holds in any coloring $[\![N]\!] \to [\![q]\!]$. There exist three numbers x, y, z, all receiving the same color, such that $x + y = z$ and x and y have a different number of digits (when written in base 10).

Solution. Let $r = r(3; q)$ and let $N = 10^r$, and fix a q-coloring $\chi : [\![N]\!] \to [\![q]\!]$. We define a coloring $\psi : E(K_r) \to [\![q]\!]$ as follows. We identify the vertex set of K_r with $[\![r]\!]$, and color an edge ab, where $a < b$, with color

$$\psi(ab) := \chi(10^b - 10^a).$$

Note that this is well-defined, since $10^b - 10^a \in [\![N]\!]$, hence this integer receives a color under χ; we assign the edge ab this same color.

By the definition of r, there is a monochromatic triangle under ψ, say with vertices a, b, c with $a < b < c$. Define $x = 10^b - 10^a, y = 10^c - 10^b$, and $z = 10^c - 10^a$, which definition implies that $x + y = z$. Moreover, since $\psi(ab) = \psi(bc) = \psi(ac)$, we have that $\chi(x) = \chi(y) = \chi(z)$. To conclude the proof, we simply note that x has exactly $b - 1$ digits in base 10, whereas y has exactly $c - 1$ digits, and $b - 1 \neq c - 1$. □

Alternate solution. By Theorem 9.3.1, we may pick N to have the following property: for every $\chi : [\![N]\!] \to [\![q]\!]$, there exist distinct $x_1, \ldots, x_{11} \in [\![N]\!]$ such that all their non-empty subset sums receive the same color under χ. We claim that this same N satisfies the desired property. Indeed, fix a coloring $\chi : [\![N]\!] \to [\![q]\!]$, and find $x_1, \ldots, x_{11}$ as above. Suppose first that for some $i, j \in [\![11]\!]$, the numbers x_i, x_j have distinct numbers of digits. Then we may set $x = x_i, y = x_j$, and $z = x_i + x_j$; these all have the same color under χ by assumption, they satisfy $x + y = z$ by definition, and x and y have distinct numbers of digits.

So we may assume that $x_1, \ldots, x_{11}$ all have the same number of digits. Let $x = x_1 + \cdots + x_{10}$, $y = x_{11}$, and $z = x + y = x_1 + \cdots + x_{11}$. We again have a monochromatic solution to $x + y = z$; the only remaining observation is that x is the sum of 10 integers with the same number of digits, so it must have one more digit, and thus has a different number of digits from y. □

6. Prove that there is an absolute constant $C > 0$ such that

$$r(k) \leqslant C \cdot r(k-1)$$

for all $k \geqslant 2$.

Solution. Let τ be the constant from Theorem 8.1.4. Let $\varepsilon > 0$ be chosen so that

$$\frac{\tau}{\varepsilon \log \frac{1}{\varepsilon}} \geqslant 4;$$

such an ε exists since the left-hand side tends to ∞ as $\varepsilon \to 0$. Finally, let $C = 6/\varepsilon$; we claim that $r(k) \leqslant C \cdot r(k-1)$ for all k.

Indeed, let $N = C \cdot r(k-1)$, and fix a 2-coloring of $E(K_N)$. It suffices to find a monochromatic K_k in this coloring. Fix a vertex $v \in V(K_N)$, and assume without loss of generality that v is incident to at least $\frac{N-1}{2} \geqslant N/3$ red edges. Let S be the set of red neighbors of S, and consider the induced coloring on S. Let G be the graph with vertex set S comprising all blue edges in S.

Suppose first that at most $\varepsilon\binom{|S|}{2}$ edges in S are blue, that is, that $d(G) \leqslant \varepsilon$. In this case, by Theorem 8.1.4, G contains a clique or an independent set of order

$$\frac{\tau}{\varepsilon \log \frac{1}{\varepsilon}} \log |S| \geqslant 4 \log \frac{N}{3} \geqslant 4 \log(r(k-1)),$$

where in the final inequality we recall that $N = C \cdot r(k-1) \geqslant 3r(k-1)$. By Theorem 2.2.2, we have that $r(k-1) \geqslant 2^{\frac{k-1}{2}}$, hence $\log(r(k-1)) \geqslant \frac{k-1}{2} \geqslant \frac{k}{4}$. Thus, G contains a clique or an independent set of order k; since G consists of the red edges in S, in either case, we have found a monochromatic K_k in the original coloring.

Therefore, we may assume that $d(G) > \varepsilon$. This means that there is some vertex $w \in S$ which is incident to at least $\varepsilon(|S|-1) \geqslant \varepsilon N/6$ blue edges in S. Let $T \subseteq S$ denote the set of blue neighbors of w in S.

That is, we have found two vertices v, w, as well as a set T, such that v is adjacent in red to all vertices of T, and w is adjacent in blue to all vertices in T. Moreover, we have that

$$|T| \geqslant \frac{\varepsilon N}{6} = \frac{\varepsilon C \cdot r(k-1)}{6} = r(k-1).$$

Therefore, in the induced coloring on T, there is a monochromatic K_{k-1}. If it is red, we may add v to it to obtain a red K_k; similarly, if it is blue, we may add w to it to obtain a blue K_k. In either case, we have found the desired monochromatic K_k. □