Homework 3

Exercise 3(c): Let n be an integer and let $0 \leqslant d \leqslant n$ be a real number. Consider a random n-vertex graph G formed by including each edge independently with probability d/n.

Prove that if $d = \omega(1)$, the average degree of G is $(1+o(1))d$ with probability $1-o(1)$.

Solution. Let X denote the number of edges of G. This is a binomial random variable with distribution $\mathrm{Bin}(\binom{n}{2}, p)$, where $p = d/n$. In particular, the expectation of X is

$$\mathbb{E}[X] = p\binom{n}{2} = \frac{d}{n} \cdot \binom{n}{2} = \frac{d(n-1)}{2} = \frac{dn}{2} - \frac{d}{2}.$$

We first claim that with probability $1-o(1)$, we have that $X = (1+o(1))dn/2$. This follows from essentially any of the standard concentration results for the binomial distribution; for concreteness, we give an elementary proof using only Chebyshev's inequality.

Since X is binomially distributed, its variance is given by

$$\mathrm{Var}(X) = p(1-p)\binom{n}{2} \leqslant p\binom{n}{2} \leqslant \frac{dn}{2}.$$

Chebyshev's inequality thus implies that for any $t > 0$, we have

$$\Pr\left(|X - \mathbb{E}[X]| \geqslant t \cdot \sqrt{\frac{dn}{2}}\right) \leqslant \frac{1}{t^2}. \tag{1}$$

We now note that

$$\begin{aligned}\Pr\left(\left|X - \frac{dn}{2}\right| \geqslant d\sqrt{\frac{n}{2}}\right) &\leqslant \Pr\left(|X - \mathbb{E}[X]| \geqslant d\sqrt{\frac{n}{2}} - \frac{d}{2}\right) \\ &\leqslant \Pr\left(|X - \mathbb{E}[X]| \geqslant \frac{d}{2}\sqrt{\frac{n}{2}}\right) \\ &= \Pr\left(|X - \mathbb{E}[X]| \geqslant \frac{\sqrt{d}}{2}\sqrt{\frac{dn}{2}}\right) \\ &\leqslant \frac{4}{d},\end{aligned}$$

where the first inequality uses the fact that $|\mathbb{E}[X] - \frac{dn}{2}| = \frac{d}{2}$, the second holds for $n \geqslant 2$ (which we are allowed to assume since we are working in the $n \to \infty$ limit), and the final holds by plugging in $t = \sqrt{d}/2$ into (1).

Recall that the average degree of any n-vertex graph G is equal to $2e(G)/n$. Hence the average degree in our random graph is $2X/n$. Let $Y = 2X/n$ be this average degree, and note that the above implies

$$\Pr\left(|Y - d| \geqslant d\sqrt{\frac{2}{n}}\right) = \Pr\left(\left|X - \frac{dn}{2}\right| \geqslant d\sqrt{\frac{n}{2}}\right) \leqslant \frac{4}{d}.$$

To conclude the proof, we note that $d\sqrt{2/n} = o(d)$ as $n \to \infty$, and that $4/d = o(1)$ since we assume $d = \omega(1)$. Hence this implies that $Y = (1+o(1))d$ with probability $1-o(1)$. $\square$

Exercise 5(a) Prove that, for any fixed $s \geqslant 3$, we have

$$r(s,k) \geqslant k^{\frac{s-1}{2}-o(1)},$$

where the $o(1)$ term tends to 0 as $k \to \infty$.

Solution. Let G be a random graph on $N := (\frac{k}{s \ln k})^{\frac{s-1}{2}}$ vertices, where each pair of vertices is included as an edge independently with probability $p := N^{-\frac{2}{s-1}}$. We begin by claiming that G is K_s-free with probability at least $\frac{5}{6}$. Indeed, any given set of s vertices forms a copy of K_s in G with probability $p^{\binom{s}{2}}$, and there are $\binom{N}{s}$ options for such a set of s vertices. Hence, by the union bound, we have that the probability that G contains a K_s is at most

$$\binom{N}{s} p^{\binom{s}{2}} \leqslant \frac{N^s}{s!} p^{\frac{s^2-s}{2}} = \frac{1}{s!}\left(Np^{\frac{s-1}{2}}\right)^s = \frac{1}{s!} \leqslant \frac{1}{6},$$

where we use our definition of p to see that $Np^{\frac{s-1}{2}} = 1$ and use the fact that $s \geqslant 3$ to conclude that $s! \geqslant 6$. Thus, G is K_s-free with probability at least $\frac{5}{6}$.

We now claim that G has no independent set of order k with probability at least $\frac{1}{2}$. Any set of k vertices forms an independent set with probability $(1-p)^{\binom{k}{2}}$, and there are $\binom{N}{k}$ choices for such a set. Applying the union bound, we find that the probability that G has an independent set of order k is at most

$$\binom{N}{k}(1-p)^{\binom{k}{2}} \leqslant \frac{N^k}{k!}(1-p)^{\frac{k^2-k}{2}} = \frac{1}{(1-p)^{\frac{k}{2}} k!} \cdot \left(N(1-p)^{\frac{k}{2}}\right)^k \tag{2}$$

Note that $p \to 0$ as $k \to \infty$, hence $p \leqslant \frac{1}{2}$ for sufficiently large k. Thus, for sufficiently large k, we have that

$$(1-p)^{\frac{k}{2}} k! \geqslant \left(\frac{1}{2}\right)^{\frac{k}{2}} k! \geqslant \left(\frac{1}{2}\right)^{\frac{k}{2}} \cdot \left(\frac{k}{2}\right)^{\frac{k}{2}} \geqslant \left(\frac{k}{4}\right)^{\frac{k}{2}} \geqslant 2,$$

where the second inequality uses the simple bound $k! \geqslant (k/2)^{k/2}$, and the final inequality also holds for sufficiently large k. On the other hand, using the bound $1-x \leqslant e^{-x}$, we have that

$$N(1-p)^{\frac{k}{2}} \leqslant Ne^{-p\frac{k}{2}} = \exp\left(\ln N - p\frac{k}{2}\right) = \exp\left(\ln N - \frac{k}{2}N^{-\frac{2}{s-1}}\right).$$

Note that, by our choice of N, we have that

$$\frac{k}{2}N^{-\frac{2}{s-1}} = \frac{k}{2} \cdot \frac{s \ln k}{k} = \frac{s \ln k}{2}$$

and that

$$\ln N \leqslant \ln\left(k^{\frac{s-1}{2}}\right) = \frac{(s-1)\ln k}{2}.$$

Therefore, $\ln N - \frac{k}{2} N^{-\frac{2}{s-1}} \leqslant -\frac{\ln k}{2} \leqslant 0$, so $N(1-p)^{\frac{k}{2}} \leqslant 1$. Plugging all of this back into (2), we find that the probability that G has an independent set of order k is at most

$$\frac{1}{(1-p)^{\frac{k}{2}} k!} \cdot \left(N(1-p)^{\frac{k}{2}}\right)^k \leqslant \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Putting this all together, we find that with probability at least $\frac{1}{2}$, G has no independent set of order k, and with probability at least $\frac{5}{6}$, G is K_s-free. Thus, with positive probability, G satisfies both these properties simultaneously, hence there exists an N-vertex graph which is K_s-free and has no independent set of order k. Therefore,

$$r(s,k) > N = \left(\frac{k}{s \ln k}\right)^{\frac{s-1}{2}} = k^{\frac{s-1}{2} - o(1)},$$

since for fixed s and for $k \to \infty$, we have that $s \ln k = k^{o(1)}$. $\square$

Homework 4

Exercise 3(a) Let q be a prime power. Construct a graph Π_q with vertex set $V(\Pi_q) = \mathbb{F}_q^2$, in which two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if and only if $x_1x_2 + y_1y_2 = 1$.

Prove that Π_q is C_4-free.

Solution. Suppose for contradiction that we have a C_4 in Π_q, namely four distinct vertices $(x_1, y_1), \ldots, (x_4, y_4)$ which are pairwise adjacent in this cyclic order. Let

$$\ell_1 := \{(x, y) \in \mathbb{F}_q^2 : xx_1 + yy_1 = 1\} \qquad \text{and} \qquad \ell_3 := \{(x, y) \in \mathbb{F}_q^2 : xx_3 + yy_3 = 1\}.$$

Note that by definition, ℓ_1 is precisely the neighborhood of (x_1, y_1) in Π_q, and similarly ℓ_3 is the neighborhood of (x_3, y_3). Moreover, by construction, ℓ_1, ℓ_3 are both lines in $\mathbb{F}_q^2$.

However, we know that two lines in $\mathbb{F}_q^2$ intersect in at most one point. Formally, suppose that $(x, y) \in \ell_1 \cap \ell_3$. Then (x, y) satisfies the two equations

$$\begin{aligned} xx_1 + yy_1 &= 1 \\ xx_3 + yy_3 &= 1. \end{aligned}$$

Subtracting the second equation from the first, we conclude that

$$x(x_3 - x_1) = y(y_1 - y_3).$$

We assumed that the points (x_1, y_1) and (x_3, y_3) were distinct, so either $x_1 \neq x_3$ or $y_1 \neq y_3$ (or both). Let us assume the first case happens; the second case is essentially identical. Since $x_1 \neq x_3$, we may divide the equation above by $x_3 - x_1$ to conclude that

$$x = \frac{y_1 - y_3}{x_3 - x_1} y. \tag{3}$$

Plugging this in to the equation $xx_1 + yy_1 = 1$, we find that

$$y\left(x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1\right) = 1.$$

Note that there is at most one choice of y satisfying this. Indeed, if $x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1 = 0$ then there is no solution to this equation, and if $x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1 \neq 0$ then the unique solution is $y = 1/(x_1 \frac{y_1 - y_3}{x_3 - x_1} + y_1)$. Plugging this back into (3) shows that, given the value of y, we can also determine the value of x.

In other words, we have proven that there is at most one point (x, y) in the intersection $\ell_1 \cap \ell_3$. Therefore, the points (x_1, y_1) and (x_3, y_3) have at most one common neighbor in Π_q, as their common neighborhood is precisely $\ell_1 \cap \ell_3$. However, our starting assumption was that (x_2, y_2) and (x_4, y_4) are distinct points, both of which are common neighbors of (x_1, y_1) and (x_3, y_3); this is a contradiction. □

Homework 6

Exercise 3(b) Let $\widehat{K_k}$ denote the 1-*subdivision* of K_k. This is a graph on $k + \binom{k}{2}$ vertices, obtained by introducing a new vertex in the middle of every edge of K_k. Equivalently, it is obtained from K_k by replacing every edge by a 2-edge path.

By applying Lemma 5.4.11 and being more careful, prove that $r(\widehat{K_k}) = O(k^2)$. Note that this bound is tight up to the implicit constant since $\widehat{K_k}$ has $\Theta(k^2)$ vertices.

Solution. By choosing the implicit constant in the big-O appropriately, we may assume that k is sufficiently large. We will assume that $k \geqslant 100$.

Let $N = 81k^2$, and fix a two-coloring of $E(K_N)$. We may assume without loss of generality that at least half the edges are red; let G be the red graph, and note that the average degree d of G satisfies $d \geqslant \frac{N-1}{2} \geqslant \frac{N}{3}$.

Let $t = \log_3 k$, let $\Delta = 2$, and let $r = k + \binom{k}{2} \leqslant k^2$. Note that

$$\begin{aligned}
\frac{d^t}{N^{t-1}} - \binom{N}{\Delta}\left(\frac{r}{N}\right)^t &\geqslant N\left(\frac{d}{N}\right)^t - N^2\left(\frac{k^2}{N}\right)^t \\
&\geqslant N\left(\frac{1}{3}\right)^t - N^2\left(\frac{1}{81}\right)^t \\
&= \frac{N}{k} - \frac{N^2}{k^4} \\
&= 81k - 81^2 \\
&\geqslant k,
\end{aligned}$$

where the final step holds since we assumed $k \geqslant 100$. Therefore, we are in the position to apply Lemma 5.4.11 with the parameters above and with $s = k$. We conclude that there is a set $T \subseteq V(G)$ of size $|T| \geqslant k$ such that every pair of vertices in T has at least $r = k + \binom{k}{2}$ common neighbors.

We now argue exactly as in the proof of Theorem 5.4.10. $\widehat{K_k}$ is a bipartite graph with one part of size k (corresponding to the original vertices of K_k) and the other of size $\binom{k}{2}$ (corresponding to the original edges of K_k). We embed the part of size k into T arbitrarily. We then arbitrarily order the vertices in the part of size $\binom{k}{2}$. Each vertex v in this part has exactly two neighbors in $\widehat{K_k}$, which were already embedded into T. By the way we constructed T, this pair of embedded vertices has at least $k + \binom{k}{2}$ common neighbors, and in particular at least one common neighbor that was not yet used in the embedding. We embed v arbitrarily into one of these common neighbors, and continuing in this process we find a red copy of of $\widehat{K_k}$. □

Exercise 5 Let G be an ε-quasirandom graph. Prove that for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geqslant \varepsilon|V(G)|$, we have $|d(S,T) - d(G)| \leqslant 3\varepsilon$.

Note: There was a typo in this homework exercise—it originally said 2ε, rather than 3ε. I'm really sorry about that!

Solution. Let S and T be disjoint sets. We begin by assuming that $|S| = |T|$; we will later get rid of this assumption. Note that every edge in $S \cup T$ is either in S, or in T, or between S and T, hence

$$e(S \cup T) = e(S) + e(T) + e(S,T).$$

If $|S| \geqslant \varepsilon|V(G)|$ and $|T| \geqslant \varepsilon|V(G)|$, then also $|S \cup T| \geqslant \varepsilon|V(G)|$. Therefore, the ε-quasirandomness of G implies that

$$d(G)-\varepsilon \leqslant d(S) \leqslant d(G)+\varepsilon, \quad d(G)-\varepsilon \leqslant d(T) \leqslant d(G)+\varepsilon, \quad d(G)-\varepsilon \leqslant d(S\cup T) \leqslant d(G)+\varepsilon.$$

Therefore,

$$e(S \cup T) \geqslant (d(G) - \varepsilon)\binom{|S \cup T|}{2}$$

and

$$e(S) \leqslant (d(G) + \varepsilon)\binom{|S|}{2}, \qquad e(T) \leqslant (d(G) + \varepsilon)\binom{|T|}{2}.$$

Combining these inequalities, we find that

$$\begin{aligned} e(S,T) &= e(S \cup T) - e(S) - e(T) \\ &\geqslant d(G)\left[\binom{|S \cup T|}{2} - \binom{|S|}{2} - \binom{|T|}{2}\right] - \varepsilon\left[\binom{|S \cup T|}{2} + \binom{|S|}{2} + \binom{|T|}{2}\right]. \end{aligned}$$

Note that $\binom{|S\cup T|}{2} - \binom{|S|}{2} - \binom{|T|}{2} = |S||T|$. This can be checked by expanding the binomial coefficients, or by a simple combinatorial argument: $\binom{|S\cup T|}{2}$ counts all pairs in $S \cup T$, and if we remove the pairs inside S and inside T, we are only left with the $|S||T|$ pairs between the sets. On the other hand, recalling that we assumed $|S| = |T|$, we have

$$\binom{|S \cup T|}{2} + \binom{|S|}{2} + \binom{|T|}{2} \leqslant \frac{1}{2}\left((|S| + |T|)^2 + |S|^2 + |T|^2\right) = 3|S|^2 = 3|S||T|.$$

Combining this with the above, we conclude that

$$e(S,T) \geqslant d(G)|S||T| - 3\varepsilon|S||T| = (d(G) - 3\varepsilon)|S||T|,$$

implying that $d(S,T) \geqslant d(G) - 3\varepsilon$. An identical computation, just swapping some plus and minus signs, shows that $d(S,T) \leqslant d(G) + 3\varepsilon$.

This concludes the proof in the case that $|S| = |T|$. In case $|S| < |T|$, we argue as follows. Let us suppose for contradiction that $d(S,T) > d(G) + 3\varepsilon$ (the other case, where $d(S,T) < d(G) - 3\varepsilon$, follows similarly). Let T' be a random subset of T, chosen uniformly at random among all subsets of size exactly $|S|$. Every vertex $v \in T$ has probability exactly $|S|/|T|$ of being included in T', hence every edge in $S \times T$ has probability of exactly $|S|/|T|$ of being included in $S \times T'$ (since all that matters is whether its T-endpoint gets included in T'). By linearity of expectation, this implies that

$$\mathbb{E}[e(S,T')] = \frac{|S|}{|T|}e(S,T) = \frac{|T'|}{|T|}e(S,T).$$

Therefore, there exists some fixed $T' \subseteq T$, with $|T'| = |S|$, such that $e(S, T') \geqslant \frac{|T'|}{|T|} e(S, T)$. Therefore,

$$d(S, T') = \frac{e(S, T')}{|S||T'|} \geqslant \frac{\frac{|T'|}{|T|} e(S, T)}{|S||T'|} = \frac{e(S, T)}{|S||T|} = d(S, T) > d(G) + 3\varepsilon.$$

However, this contradicts our previous argument, since $|S| = |T|'$, so we know that $d(S, T') \leqslant d(G) + 3\varepsilon$. □

Exercise 7 Prove Theorem 6.2.3, the linear bound on multicolor Ramsey numbers of bounded-degree graphs.

Solution. First we pick some parameters depending on Δ and q. Let $\varepsilon = q^{-\Delta}/(2\Delta)$, which is chosen so that $\frac{1}{q} = (2\Delta\varepsilon)^{1/\Delta}$. Let $\delta(\varepsilon, q)$ be the constant from Lemma 6.2.1. Finally, let $C = 2/(\varepsilon\delta)$, and note that C depends only on Δ and q.

Fix an n-vertex graph H with maximum degree at most Δ, and let $N = Cn$. Consider a q-coloring of $E(K_N)$, and let $G_1, \ldots, G_q$ be the q color classes. Applying Lemma 6.2.1, we find a subset $Q \subseteq V(K_N)$ with $|Q| \geqslant \delta N$ such that $G_1[Q], \ldots, G_q[Q]$ are all ε-quasirandom. By the pigeonhole principle, among the edges in Q, at least a $\frac{1}{q}$ fraction have the same color. So we may pick some $i \in [\![q]\!]$ such that at least $\frac{1}{q}\binom{|Q|}{2}$ of the edges in Q have color i; equivalently, this says that $d(G_i[Q]) \geqslant \frac{1}{q} = (2\Delta\varepsilon)^{1/\Delta}$. Note that

$$|Q| \geqslant \delta N = \delta C n = \frac{2n}{\varepsilon}.$$

Thus, we are in the setting of Lemma 6.1.3, which immediately tells us that H is a subgraph of $G_i[Q]$. Thus, we have found a monochromatic copy of H in color i, implying that $r(H) \leqslant N$. □

Homework 7

Exercise 1

(a) Fix a q-coloring $\chi_0 : E(K_n) \to [\![q]\!]$. Prove that for every $\sigma > 0$, there exists $\delta > 0$ such that the following holds. If a coloring $\chi : E(K_N) \to [\![q]\!]$ does not contain χ_0 as an induced subcoloring, then there exists a set $S \subseteq V(K_N)$ with $|S| \geqslant \delta N$ and an index $i \in [\![q]\!]$, such that at most $\sigma\binom{|S|}{2}$ of the edges in S are colored by color i under χ.

(b) Prove that the $q = 2$ case of part (a) is equivalent to Rödl's theorem, Theorem 6.3.3.

(c) You might have expected the multicolor generalization of Rödl's theorem to say that *all colors but one* have edge density at most σ in S. Prove that such a statement is false, even in the case $n = q = 3$.

More precisely, show that there is a $E(K_N) \to [\![3]\!]$ such that no triangle receives all three colors, but such that every linearly-sized subset has edge density at least $\frac{1}{3}$ in at least two of the colors.

Solution.

(a) Fix χ_0 and $\sigma > 0$. Let $\varepsilon = \sigma^n/(2n)$, and let $\delta_0 > 0$ be the parameter from Lemma 6.2.1 applied with this choice of ε and q. Let $\delta = \varepsilon\delta_0/(2n)$.

Now fix $\chi : E(K_N) \to [\![q]\!]$, and let $G_1, \ldots, G_q$ be the graphs of the edges in colors $1, \ldots, q$, respectively. If $N \leqslant \frac{1}{\delta}$ then we are done, since we may set S to be a single vertex, so we assume henceforth that $N \geqslant \frac{1}{\delta}$. We apply Lemma 6.2.1 to χ to find a set $Q \subseteq V(K_N)$ with $|Q| \geqslant \delta N$ such that all of the graphs $G_1[Q], \ldots, G_q[Q]$ are ε-quasirandom. If $d(G_i[Q]) < \sigma$ for some i, we are done: we may set $S = Q$, and then we know that at most $\sigma\binom{|S|}{2}$ edges in S are colored with color i. So we may assume that $d(G_i[Q]) \geqslant \sigma$ for all i. But by our choice of ε and our assumption that $N \geqslant 1/\delta$, we may now apply Lemma 6.3.4 to conclude that there is a copy of K_n in K_N which is colored according to χ_0. This contradiction completes the proof.

(b) A 2-coloring χ_0 of $E(K_n)$ is the same as an n-vertex graph H—we just view the red edges as H and the blue edges as the complement of H. Moreover, a copy of χ_0 inside a coloring of $E(K_N)$ is the same as finding H as an induced subgraph of the N-vertex graph G, since being a copy of χ_0 means that edges of H yield edges of G, and non-edges of H yield non-edges of G. Moreover, since there are only two colors, saying that one of the colors has density at most σ is precisely the same as saying that $d(G[S]) \leqslant \sigma$ or $d(G[S]) \geqslant 1 - \sigma$.

(c) Consider a random coloring of $E(K_N)$, where each edge is made red or blue with probability $\frac{1}{2}$. We can view this as a 3-coloring of $E(K_N)$, where no edge receives the color green. In particular, this coloring has no copy of a colored K_3, where all three edges are colored green. Therefore we are in the setting of part (a), but with high probability, every subset $S \subseteq V(K_N)$ of size δN has at least $\frac{1}{3}\binom{|S|}{2}$ red and at least $\frac{1}{3}\binom{|S|}{2}$ blue edges. Thus we cannot ensure that all but one color has very low density. □

Exercise 3(a) A graph H is said to have the *Erdős–Hajnal property* if there exists $\varepsilon > 0$, depending only on H, such that every induced-H-free N-vertex graph has a clique or an independent set of size at least N^ε. Recall that the Erdős–Hajnal conjecture asserts that all graphs have the Erdős–Hajnal property.

Prove that if $H = K_k$ is a complete graph, then H has the Erdős–Hajnal property.

Solution. By Theorem 2.1.4, the off-diagonal Ramsey number $r(k, t)$ satisfies $r(k, t) \leqslant \binom{k+t}{k}$. Note that, if $k \leqslant t$, we have

$$r(k, t) \leqslant \binom{k+t}{k} \leqslant \binom{2t}{k} \leqslant (2t)^k \leqslant t^{2k}.$$

On the other hand, if $2 \leqslant t \leqslant k$, then

$$r(k,t) \leqslant \binom{k+t}{k} \leqslant \binom{2k}{k} \leqslant 4^k = 2^{2k} \leqslant t^{2k},$$

so in either case we have $r(k,t) \leqslant t^{2k}$.

We now claim that K_k has the Erdős–Hajnal property with parameter $\varepsilon = 1/(2k)$. Indeed, let G be an N-vertex graph which is induced-K_k-free. Let $t = N^{1/(2k)}$. By the above, we have that $r(k,t) \leqslant t^{2k} = N$, meaning that G contains a K_k or an independent set of size t, and by the assumption it does not contain the former. This shows that G contains an independent set of size $t = N^{1/(2k)} = N^{\varepsilon}$, as claimed. □

Exercise 5 A graph G is called *q-minimally Ramsey* for a graph H if G is Ramsey for H in q colors, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H in q colors.

(a) Prove that if G is q-minimally Ramsey for H, then every edge of G lies in at least q copies of H.

(b) Prove that if G is q-minimally Ramsey for H, then G has at least $q^{e(H)-1}$ copies of H.

(c) Prove Proposition 7.1.9.

Solution.

(a) Let G be q-minimally Ramsey for H, and fix some edge $e \in E(G)$. Consider the graph $G - e$. Since G is minimally Ramsey for H, there exists a coloring $\chi : E(G-e) \to [\![q]\!]$ with no monochromatic copy of H. For each $i \in [\![q]\!]$, let $\chi_i : E(G) \to [\![q]\!]$ be the coloring of $E(G)$ obtained from i by coloring all edges other than e the same as in χ, and coloring e in color i. Since G is Ramsey for H, there must exist a monochromatic copy of H in each of the q colorings χ_i. Moreover, since e gets a different color in each of these colorings, we must get q distinct copies of H, each containing e, as claimed.

(b) Suppose that G is a graph with fewer than $q^{e(H)-1}$ copies of H. Let $\chi : E(G) \to [\![q]\!]$ be a uniformly random q-coloring of $E(G)$. For each copy of H in G, the probability that it is monochromatic under χ is exactly $q^{1-e(H)}$. By the union bound, the probability that there is no monochromatic copy of H is therefore at most $q^{1-e(H)}$ times the number of copies of H in G, which is strictly less than 1 by assumption. Hence, there exists a q-coloring of $E(G)$ with no monochromatic H, showing that G is not Ramsey for H.

(c) Note that if F is Ramsey obligatory for K_3, then so is any subgraph of it, hence it suffices to prove that if F is a triangle tree, then F is Ramsey obligatory for K_3. We prove this by induction on the number of triangles forming F. In fact, we prove the following stronger statement: if F is a triangle tree composed of t triangles, then $F \subseteq G$ for every graph which is $(3t)$-color Ramsey for K_3. In particular, this implies that every triangle tree is Ramsey obligatory for K_3.

The base case is when $t = 1$, and thus $F = K_3$. Any graph which is 3-color Ramsey for K_3 must contain K_3, hence the base case is proved. Inductively, suppose we have proved the statement for t, and let F be a triangle tree composed of $t+1$ triangles. By definition, F is obtained from some triangle tree F' by adding a new triangle on some edge $e \in E(F')$, where F' is composed of t triangles.

Now, let G be a graph which is $3(t+1)$-color Ramsey for K_3. In particular, it is $3t$-color Ramsey for K_3, so by the induction hypothesis there is a copy of F' in G. Moreover, by part (a), the edge $e \in E(F') \subseteq E(G)$ lies in at least $3(t+1)$ triangles in G. Since F' has at most $3t$ vertices, at least one of these triangles containing e must not use any other vertices of F'. Hence we can extend the copy of F' to a copy of F by adding one of these triangles along e, proving the statement for $t+1$ and completing the induction. □

Exercise 6 Prove that for every $n, q \geqslant 2$, there exists some N such that $K_{N,N}$ is q-color induced Ramsey for $K_{n,n}$.

Solution. Pick N sufficiently large so that

$$\frac{N^2}{q} \geqslant n^{\frac{1}{n}}(2N)^{2-\frac{1}{n}} + 2nN.$$

Note that this inequality is satisfied for sufficiently large N (and fixed n, q), since the left-hand side grows as N^2 and the right-hand side grows as $O(N^{2-1/n})$.

Now, fix a q-coloring of $E(K_{N,N})$. One of the colors must contain at least N^2/q edges. Let G be the graph of the edges in this color, which has $2N$ vertices and at least N^2/q edges. By the choice of N and Theorem 5.3.2, we conclude that $K_{n,n} \subseteq G$, hence there is a monochromatic copy of $K_{n,n}$ in the coloring. Moreover, every copy of $K_{n,n}$ in $K_{N,N}$ is an induced copy, so we conclude that $K_{N,N}$ is induced Ramsey for $K_{n,n}$ in q colors. □

Homework 8

Exercise 3 Let $1 \leqslant \ell \leqslant q-1$ be integers, and let $\binom{[\![q]\!]}{\ell}$ denote the collection of all ℓ-element subsets of $[\![q]\!]$. A (q,ℓ)-*set coloring* is a function $\chi : E(K_N) \to \binom{[\![q]\!]}{\ell}$; in other words, rather than assigning every edge of K_N a single color out of q options, we assign every edge a list of ℓ colors from a palette of size q. We say that $v_1, \dots, v_k \in V(K_N)$ form a *color-intersecting clique* if there is a color that appears in all of the $\binom{k}{2}$ lists associated to the edges they span, that is, if $\bigcap_{1 \leqslant i < j \leqslant k} \chi(v_i v_j) \neq \varnothing$. The *set coloring Ramsey number* $r_s(k; (q,\ell))$ is the least N such that every (q,ℓ)-set coloring of $E(K_N)$ contains a color-intersecting clique of order k.

(e) Prove that, for every $\varepsilon > 0$ there exists some $B > 0$ such that the following holds. If $\ell \geqslant \varepsilon q$, then $r_s(k; (q,\ell)) \leqslant 2^{Bkq}$.

(f) Using Theorem 8.1.4, prove the following. For every $x \geqslant 1$, there exists $D > 0$ such that
$$r_s(k; (q, q-x)) \leqslant 2^{\frac{Dk}{q} \log q}.$$
Note that this bound is much stronger than that given in (e).

Solution.

(e) Consider a (q, ℓ)-set coloring of $E(K_N)$, where $\ell \geqslant \varepsilon q$. For a fixed vertex v, there are $N-1$ edges incident to v, hence a total of $\ell(N-1)$ colors used on its edges. However, since the total palette has only q colors, there must exist some color which appears on at least $\lceil \ell(N-1)/q \rceil \geqslant \frac{\varepsilon}{2} N$ of the edges incident to v, where the last inequality holds as long as $N \geqslant \frac{1}{\varepsilon}$.

Now, let $N = 2^{Bkq}$, where $B = \frac{2}{\varepsilon}$. We repeatedly apply the observation above, as follows. We start with some vertex v_1, and find a color c_1 that lies on at least $\frac{\varepsilon}{2} N$ of the edges incident to v_1. We now restrict to this c_1-colored neighborhood, find a new vertex v_2 and a new color c_2, and so on. By our choice of N, we can continue this process for at least kq steps, since we lose a factor of $\varepsilon/2$ at every step. At the end of the process, at least k of these kq colors must be equal, hence we find a color-intersecting K_k. □

Exercise 4

(a) Prove that Theorem 8.2.4 is equivalent to the following statement. For every $C > 0, k \in \mathbb{N}$, the following holds for sufficiently large N. Consider a coloring $\chi : E(K_N) \to \{\text{red, blue}\}$, and suppose that χ contains no monochromatic clique of order $C \log N$. Then for every coloring $\psi : E(K_k) \to \{\text{red, blue}\}$, there is a k-vertex subset S of K_N such that the restriction of χ to S equals ψ (up to permutations of the vertices).

(b) State and prove a generalization of (a) to colorings with more than two colors.

Solution.

(a) A red/blue coloring of a graph is the same as a graph. In particular, χ containing no monochromatic clique of order $C \log N$ is the same as saying that the graph G of red edges contains no clique or independent set of order $C \log N$, that is, that G is C-Ramsey. So the assumptions of the two statements are equivalent. The conclusions are equivalent too, since containing a copy of ψ is the same as having the red graph H of ψ as an induced subgraph.

(b) The general statement is as follows. For every $C > 0, k, q \in \mathbb{N}$, the following holds for sufficiently large N. If a q-coloring $\chi : E(K_N) \to [\![q]\!]$ contains no subset of order $C \log N$ which is colored with at most $q-1$ colors, then for every $\psi : E(K_k) \to [\![q]\!]$, there is a k-vertex subset $S \subseteq V(K_N)$ with $\chi|_S = \psi$.

To prove this, let $G_1, \ldots, G_q$ be the graphs of edges in each of the q colors, and fix some $\psi : E(K_k) \to [\![q]\!]$. Our assumption implies that each G_i is C-Ramsey; indeed, if some G_i had a clique or an independent set of order $C \log N$, this would yield a set of size $C \log N$ which uses at most $q-1$ colors. By Theorem 8.2.3, there is some $\sigma > 0$ such that every $S \subseteq V(K_N)$ with $|S| \geqslant \sqrt{N}$ satisfies $\sigma \leqslant d(G_i[S]) \leqslant 1-\sigma$ for every $i \in [\![q]\!]$. Moreover, by exercise 1(a) on Homework 7, we know if that if χ does not contain ψ as an induced sub-coloring, then there is some $S \subseteq V(K_N)$ with $|S| \geqslant \delta N$ such that, for some $i \in [\![q]\!]$, we have $d(G_i[S]) < \sigma$. But if we pick N sufficiently large so that $\delta N \geqslant \sqrt{N}$, this is a contradiction. □

Homework 9

Exercise 2

(a) Prove that, for every $k, q \geqslant 2$, there exists some N such that any q-coloring of $[\![N]\!]$ contains a k-term *geometric* progression. That is, there exist numbers a, r with $r \geqslant 2$ such that

$$a, ar, ar^2, \ldots, ar^{k-1}$$

all receive the same color.

Hint: This is a one-line corollary of van der Waerden's theorem.

(b) Prove the following multiplicative analogue of Theorem 9.3.1. For every $m, q \geqslant 2$, there exists N such that in any q-coloring of $[\![N]\!]$, there exist distinct $x_1, \ldots, x_m \in [\![N]\!]$ such that all the subset products $\prod_{i \in I} x_i$, for $\varnothing \neq I \subseteq [\![m]\!]$, receive the same color.

Solution.

(a) Let $N = 2^{W(k;q)}$, and consider a q-coloring $\chi : [\![N]\!] \to [\![q]\!]$. Define $\chi' : [\![W(k;q)]\!] \to [\![q]\!]$ by $\chi'(x) = \chi(2^x)$, and note that this is well-defined since $2^x \in [\![N]\!]$ for all $x \in [\![W(k;q)]\!]$, by our choice of N. By the definition of $W(k;q)$, there is a monochromatic k-AP under χ', say

$$\chi'(a_0) = \chi'(a_0 + r_0) = \cdots = \chi'(a_0 + (k-1)r_0).$$

Recalling the definition of χ', this implies that

$$\chi(2^{a_0}) = \chi(2^{a_0+r_0}) = \cdots = \chi(2^{a_0+(k-1)r_0}).$$

Letting $a = 2^{a_0}$ and $r = 2^{r_0}$, this is the same as saying

$$\chi(a) = \chi(ar) = \cdots = \chi(ar^{k-1}),$$

and thus we have found a monochromatic k-term geometric progression under χ.

(b) The proof is very similar. Let N_0 be sufficiently large so that Theorem 9.3.1 holds, that is, so that any q-coloring of $[\![N_0]\!]$ contains a set of size m all of whose subset sums have the same color. Let $N = 2^{N_0}$, and fix a coloring $\chi : [\![N]\!] \to [\![q]\!]$. This defines a coloring $\chi' : [\![N_0]\!] \to [\![q]\!]$ by $\chi'(x) = \chi(2^x)$. By the choice of N_0 we obtain $x'_1, \ldots, x'_m$ all of whose subset sums receive the same color under χ', meaning that if we set $x_i = 2^{x'_i}$ we obtain the desired conclusion. □

Exercise 4(b) Prove that there are at most $N^2/(2(k-1))$ arithmetic progressions of length k in $[\![N]\!]$. Using this, prove that

$$W(k;q) > \sqrt{2(k-1)}q^{\frac{k-1}{2}}.$$

Solution. For $x \in [\![N]\!]$, let $f(x)$ denote the number of k-APs in $[\![N]\!]$ whose last element is x. Then the total number of k-APs in $[\![N]\!]$ is just $\sum_{x\in[\![N]\!]} f(x)$. Consider a k-AP ending at x, say that this k-AP is $a, a+r, \ldots, a+(k-1)r = x$. Since $a \geqslant 1$, we conclude that

$$x = a + (k-1)r \geqslant 1 + (k-1)r,$$

which implies that $r \leqslant (x-1)/(k-1)$. Moreover, once we fix x as the last element of a k-AP, as well as fixing the common difference, we have completely determined the k-AP. We thus conclude that $f(x) \leqslant (x-1)/(k-1)$. Therefore, the total number of k-APs in $[\![N]\!]$ is at most

$$\sum_{x=1}^{N} f(x) \leqslant \sum_{x=1}^{N} \frac{x-1}{k-1} = \frac{1}{k-1}\sum_{x=1}^{N}(x-1) = \frac{1}{k-1}\binom{N}{2} < \frac{N^2}{2(k-1)}.$$

Now let $N = \sqrt{2(k-1)}q^{\frac{k-1}{2}}$. To deduce a lower bound on $W(k;q)$, we consider a uniformly random q-coloring of $[\![N]\!]$. For any fixed k-AP in $[\![N]\!]$, the probability that it is monochromatic is exactly q^{1-k}, since there are k elements that must receive the same color and q choices for this color. By the union bound, the probability that some k-AP is monochromatic is at most q^{1-k} times the number of k-APs, which by the above is strictly less than

$$q^{1-k} \cdot \frac{N^2}{2(k-1)} = q^{1-k} \cdot \frac{2(k-1)q^{k-1}}{2(k-1)} = 1.$$

Hence with positive probability there is no monochromatic k-AP in this random coloring, implying that $W(k;q) > N$. □

Homework 10

Exercise 1 A function $\varphi : [\![k]\!]^s \to [\![k]\!]^d$ is the same as a tuple $\varphi = (\varphi_1, \ldots, \varphi_d)$ of functions $\varphi_j : [\![k]\!]^s \to [\![k]\!]$. Such a function φ is called a *combinatorial mapping* if every component φ_j is either a constant function or a coordinate function, i.e. $\varphi_j(x_1, \ldots, x_s) = x_i$ for some i. An *s-dimensional combinatorial subspace* of $[\![k]\!]^d$ is the image of a combinatorial mapping $\varphi : [\![k]\!]^s \to [\![k]\!]^d$ which is furthermore injective.

(b) Show that s-dimensional combinatorial subspaces of $[\![k]\!]^d$ are in bijection with *s-roots*, which are words $\rho \in \{1, \dots, k, *_1, \dots, *_s\}^d$ in which each star symbol $*_i$ appears at least once.

(c) Prove that for every $k, s, q \geqslant 1$, there exists some d such that any q-coloring of $[\![k]\!]^d$ contains a monochromatic s-dimensional combinatorial subspace.

Hint: Prove that $d = s \cdot \mathrm{HJ}(k^s; q)$ suffices.

Solution.

(b) Let $\varphi = (\varphi_1, \dots, \varphi_d) : [\![k]\!]^s \to [\![k]\!]^d$ be a combinatorial mapping. Define an s-root ρ as follows. If φ_j is a constant function, say $\varphi_j(x) = a_j \in [\![k]\!]$, then let the jth coordinate of ρ be a_j. On the other hand, if φ_j is a coordinate function x_i, set the jth coordinate of ρ be $*_i$. Note that since ϕ is injective, every coordinate function x_i must appear as some coordinate of φ, hence every star symbol is used at least once, so ρ is a valid s-root. Moreover, this process is reversible: given a root ρ we construct φ by simply undoing the operations above. This shows the bijection between combinatorial mappings and s-roots. Finally, there is a bijection between combinatorial mappings and their images, proving the desired result.

(c) Let $d_0 = \mathrm{HJ}(k^s; q)$ and $d = s \cdot d_0$, and consider a coloring $\chi : [\![k]\!]^d \to [\![q]\!]$. We may identify $[\![k]\!]^d$ with $([\![k]\!]^s)^{d_0}$. Moreover, we may identify $[\![k]\!]^s$ with $[\![k^s]\!]$. By the choice of d_0, there is a monochromatic combinatorial line under χ, where a combinatorial line is with respect to the alphabet $[\![k]\!]^s$, that is, it has length k^s and the moving coordinates take on each of the elements of $[\![k]\!]^s$ in turn. But this is nothing more than (a special kind of) s-dimensional combinatorial subspace, implying that we have a monochromatic s-dimensional combinatorial subspace. □

Exercise 2

(a) Suppose that there is a coloring $\chi : [\![N]\!]^t \to [\![q]\!]$ with no homothetic copy of

$$S \coloneqq \{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}.$$

Using χ, construct a protocol for t players to compute the exactly-N function using at most $t\lceil \log q \rceil$ bits of communication in the number-on-the-forehead model.

(b) Reinterpret the result of (a) as saying the following: If the Gallai–Witt theorem is false for this choice of S, then there *is* a protocol to compute the exactly-N function using only a constant number of bits of communication.

In other words, we proved in Theorem 9.4.1 that the Gallai–Witt theorem implies a super-constant lower bound for this communication complexity, and (a) gives a converse: a super-constant lower bound for this communication complexity implies the Gallai–Witt theorem for this choice of S.

(c) Improve your protocol in (a) to one using only $t + \lceil \log q \rceil$ bits of communication.

Solution.

(a) Let the players receive input $a = (a_1, \ldots, a_t) \in [\![N]\!]^t$, where player i sees all the coordinates except for a_i. Using this information, player i computes the value

$$b_i := N - a_1 - \cdots - a_{i-1} - a_{i+1} - \cdots - a_t.$$

That is, b_i is the unique possible value of the ith coordinate that makes the exactly-N function evaluate to 1.

Now, for every $i \in [\![t]\!]$, the ith player communicates the value of

$$\chi((a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_t)).$$

Since there are q colors, it takes $\lceil \log q \rceil$ bits to communicate this value, and since all t players do it this costs a total of $t\lceil \log q \rceil$ bits of communcation. Finally, the players output the value 1 if all t players communicated the same thing, and the output 0 otherwise.

We claim that this is a valid protocol. First of all, note that if $\sum a_i = N$ (in which case the players are supposed to output 1), we have $b_i = a_i$ for all i. Hence all t players simply say $\chi(a)$, and in particular they all agree, so they do indeed output the correct answer.

The remaining case is if $\sum a_i \neq N$. In this case, the points

$$\{(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_t) : i \in [\![t]\!]\}$$

form a homothetic copy of S. So by the assumption on χ, it must be that not all of these points receive the same color under χ. That is, the things said by the t players will not all be equal, so they will correctly output the answer 0.

(b) If the Gallai–Witt theorem were false for this choice of S, then that means that there exists some q such that for all N, there exists a q-coloring $[\![N]\!]^t \to [\![q]\!]$ with no homothetic copy of S. But by part (a), this means that one can compute the exactly-N function using only $t\lceil \log q \rceil$ bits, regardless of the value of N. Since t and q are independent of N, this shows that there is a constant-complexity communication protocol for the exactly-N function.

(c) It is wasteful for all t players to say the full value of $\chi((a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_t))$. Instead, the first player could communicate this (i.e. with $i = 1$), and then each subsequent player can say whether that agrees with what they would have said. Thus the first player communicates $\lceil \log q \rceil$ bits, and all subsequent players only communicate one bit, leading to a total complexity of $(t-1) + \lceil \log q \rceil$. At the end of the process, the players still know whether to output 0 or 1, for the same reason as in (a). $\square$

Exercise 3 Prove the density Hales–Jewett theorem for $k = 2$. In other words, prove that for every $\delta > 0$ and every sufficiently large d, every subset $A \subseteq [\![2]\!]^d$ with $|A| \geqslant \delta 2^d$ contains a combinatorial line.

Solution. We will actually show that if $A \subseteq [\![2]\!]^d$ satisfies $|A| > \binom{d}{\lfloor d/2 \rfloor}$, then A contains a combinatorial line. This is a result known as Sperner's theorem, but we will give a short proof. Note that this implies the desired result, since by Stirling's approximation we have that

$$\binom{d}{\lfloor d/2 \rfloor} = O\left(\frac{1}{\sqrt{d}} 2^d\right).$$

In particular, if d is sufficiently large with respect to δ, then we have that $\binom{d}{\lfloor d/2 \rfloor} < \delta 2^d$, implying the desired result.

Fix a set $A \subseteq [\![2]\!]^d$, and suppose that A contains no combinatorial line. We pick a random permutation π of $[\![d]\!]$, and define a sequence of vectors $x^{(0)}, \ldots, x^{(d)} \in [\![2]\!]^d$ as follows. The ith vector $x^{(i)}$ has a 1 in the entries $\{\pi(1), \pi(2), \ldots, \pi(i)\}$, and a 0 in all other entries. In other words, $x^{(0)}$ is the all 0s vector, $x^{(1)}$ is obtained from $x^{(0)}$ by flipping a single random entry to a 1, $x^{(2)}$ is obtained by flipping a new random entry to a 1, and so on, all the way up to $x^{(d)}$ being the all-1s vector.

Note that every pair $x^{(i)}, x^{(j)}$ with $i < j$ forms a combinatorial line. Indeed, $x^{(j)}$ is obtained from $x^{(i)}$ by flipping some set of the coordinates from 0 to 1. So if we make these the moving coordinates and all other coordinates constant, we see that we have a combinatorial line. In particular, since we assume that A has no combinatorial line, we must have that $x^{(i)} \in A$ for at most one i. In particular,

$$\sum_{i=0}^{d} \Pr(x^{(i)} \in A) \leqslant 1,$$

since these events are mutually exclusive.

For $0 \leqslant i \leqslant d$, let A_i be the set of elements of A with exactly i entries equal to 1. If we just consider $x^{(i)}$, it is a random vector, chosen uniformly at random among those vectors with exactly i ones. That is,

$$\Pr(x^{(i)} \in A) = \Pr(x^{(i)} \in A_i) = \frac{|A_i|}{\binom{d}{i}} \geqslant \frac{|A_i|}{\binom{d}{\lfloor d/2 \rfloor}},$$

where in the final step we used the fact that the central binomial coefficient $\binom{d}{\lfloor d/2 \rfloor}$ is the largest binomial coefficient in the dth row of Pascal's triangle. Combining these two inequalities, we find that

$$1 \geqslant \sum_{i=0}^{d} \Pr(x^{(i)} \in A) \geqslant \sum_{i=0}^{d} \frac{|A_i|}{\binom{d}{\lfloor d/2 \rfloor}} = \frac{1}{\binom{d}{\lfloor d/2 \rfloor}} \sum_{i=0}^{d} |A_i| = \frac{|A|}{\binom{d}{\lfloor d/2 \rfloor}}.$$

Rearranging shows that $|A| \leqslant \binom{d}{\lfloor d/2 \rfloor}$, yielding the desired result. □

Exercise 5 Let us say that a graph H has the *density Ramsey property* if for every $\delta > 0$ and every sufficiently large N, any N-vertex graph G with at least $\delta\binom{N}{2}$ edges has a copy of H.

(b) Prove that if H is bipartite, then H has the density Ramsey property.

(c) Prove that if H is not bipartite, then H does not have the density Ramsey property.

Solution.

(b) Let H be a bipartite graph with parts of size s, t, where $s \leqslant t$. This means that H is a subgraph of $K_{s,t}$. By Theorem 5.3.2, if G is an N-vertex graph with at least $t^{\frac{1}{s}}N^{2-\frac{1}{s}} + sN$ edges, then $K_{s,t} \subseteq G$. Note that if δ, s, t are fixed and N is sufficiently large, then

$$\delta\binom{N}{2} > t^{\frac{1}{s}}N^{2-\frac{1}{s}} + sN,$$

since the left-hand side grows quadratically in N and the right-hand side grows as $O(N^{2-\frac{1}{s}})$. This shows that if N is sufficiently large, every graph G with at least $\delta\binom{N}{2}$ edges contains a copy of $K_{s,t}$, and in particular a copy of $H \subseteq K_{s,t}$.

(c) Let H be a non-bipartite graph. For any N, let G be the complete bipartite graph with parts of sizes $\lfloor N/2 \rfloor, \lceil N/2 \rceil$, which has at least $\frac{1}{2}\binom{N}{2}$ edges. Since H is not bipartite, G has no copy of H. This shows that the density Ramsey property fails for H, with $\delta = \frac{1}{2}$. □

Exercise 6(a) The *finite unions theorem* states the following. For every $m, q \geqslant 2$, there exists some N such that in any q-coloring of $2^{[\![N]\!]}$ (that is, every subset of $[\![N]\!]$ receives some color), there exist disjoint sets $S_1, \ldots, S_m \subseteq [\![N]\!]$ such that all of the unions $\bigcup_{i \in I} S_i$, for $\varnothing \neq I \subseteq [\![m]\!]$, receive the same color.

Prove that the finite unions theorem implies Theorem 9.3.1.

Solution. Let N_0 be such that every q-coloring of $2^{[\![N_0]\!]}$ contains m disjoint sets such that all the unions receive the same color. Let $N = 2^{N_0}$, and fix a q-coloring $\chi : [\![N]\!] \to [\![q]\!]$. We define a q-coloring $\chi' : 2^{[\![N_0]\!]} \to [\![q]\!]$ as follows. Given a set $S \in 2^{[\![N_0]\!]}$, we may write the *indicator vector* of S, which is a vector in $\{0,1\}^{N_0}$ whose ith coordinate is 1 if $i \in S$, and 0 otherwise. We may then interpret this indicator vector as a number written in binary, to obtain an integer $n_S \in [\![N]\!]$. We then define $\chi'(S) := \chi(n_S)$.

By the choice of N_0, there exist disjoint $S_1, \ldots, S_m \in 2^{[\![N_0]\!]}$ such that all of their subset unions receive the same color. Note that since $S_1, \ldots, S_m$ are disjoint, we have that $n_{S_1 \cup S_2} = n_{S_1} + n_{S_2}$, since if we do the addition on the right-hand side in base 2, we are precisely summing up the indicator vectors of S_1 and S_2, with no carries since the sets are disjoint. More generally, for every $\varnothing \neq I \subseteq [\![m]\!]$, we have

$$n_{\bigcup_{i \in I} S_i} = \sum_{i \in I} n_{S_i}.$$

By the choice of χ', this implies that all the non-empty subset sums of $n_{S_1}, \ldots, n_{S_m}$ receive the same color under χ, proving Theorem 9.3.1. $\square$

Exercise 7 For a bipartite graph H and a number $\delta > 0$, let $r_d(H; \delta)$ denote the minimum integer N such that every N-vertex graph with at least $\delta\binom{N}{2}$ edges has a copy of H. (Note that this is a well-defined quantity, by problem 5(b).)

(a) By examining your solution to problem 5(b), show that for every bipartite graph H, there exists some $C > 0$ such that

$$r_d(H; \delta) \leqslant \left(\frac{1}{\delta}\right)^C$$

for all $0 < \delta \leqslant \frac{1}{2}$.

(b) Let H be a graph, and suppose G is an N-vertex graph with $\delta\binom{N}{2}$ edges and with no copy of H. Prove that if q is an integer satisfying $(1-\delta)^q\binom{N}{2} < 1$, then

$$r(H; q) > N.$$

Hint: Randomly permute the vertices of G to obtain q copies $G_1, \ldots, G_q$. Show that with positive probability, every edge of K_N appears in at least one G_i.

(c) Fix a bipartite graph H, and let C be the constant from part (a). Show that

$$r_d\left(H; \frac{2C \ln q}{q}\right) \leqslant r(H; q) \leqslant r_d\left(H; \frac{1}{q}\right),$$

where the lower bound uses part (b) and the upper bound uses your solution to problem 5(a). This shows that $r(H; q)$ and $r_d(H; 1/q)$ are closely related for bipartite H.

Solution.

(a) Recall that our proof of exercise 5(b) showed that $r_d(H; \delta)$ is at most the least N such that

$$\delta\binom{N}{2} > t^{\frac{1}{s}} N^{2-\frac{1}{s}} + sN,$$

where H has parts of sizes $s \leqslant t$. Solving this inequality for N shows that it holds when $N = O((\frac{1}{\delta})^s)$. By defining $C \geqslant s$ sufficiently large to absorb the big-O term in the exponent, we get the claimed result.

(b) Let G' be a random copy of G, obtained by randomly permuting the vertices of G. For every fixed edge $e \in E(K_N)$, the probability that $e \in E(G')$ is exaclty δ, since each of the $\binom{N}{2}$ edges of K_N are equally likely to appear in G', and there are $\delta\binom{N}{2}$ edges of G'.

Now, let $G_1, \ldots, G_q$ be independently random copies of G. By the above, for any fixed edge $e \in E(K_N)$, we have that $\Pr(e \notin E(G_i)) = 1 - \delta$. Since $G_1, \ldots, G_q$ are independent, this implies that $\Pr(e \notin E(G_1) \cup \cdots \cup E(G_q)) = (1-\delta)^q$. By the union bound, the probability that some edge of K_N does not appear in $G_1, \ldots, G_q$ is at most $(1-\delta)^q \binom{N}{2}$, which is less than 1 by assumption.

Therefore, there exist some $G_1, \ldots, G_q$, each a copy of G, whose edges cover all edges of K_N. We may now define a q-coloring of $E(K_N)$, by coloring an edge e according to the first index $i \in \llbracket q \rrbracket$ such that $e \in E(G_i)$. Note that the ith color class is a subgraph of G_i, so since $G_i \cong G$ is H-free, there is no monochromatic copy of H in this coloring. This shows that $r(H; q) > N$.

(c) The upper bound $r(H;q) \leqslant r_d(H; \frac{1}{q})$ is immediate, since if $N = r_d(H; \frac{1}{q})$, then in any q-coloring of $E(K_N)$, one of the color classes must have at least $\frac{1}{q}\binom{N}{2}$ edges, and hence we obtain a monochromatic copy of H. For the lower bound, let $\delta = \frac{2C \ln q}{q}$ and $N = r_d(H;\delta) - 1$. By definition, there is a graph G on N vertices with at least $\delta\binom{N}{2}$ edges and no copy of H. Note that

$$(1-\delta)^q \binom{N}{2} < e^{-\delta q} N^2 = q^{-2C} N^2 \leqslant q^{-2C} \delta^{-2C} = \left(\frac{1}{q\delta}\right)^{2C},$$

where the final inequality uses that $N < r_d(H;\delta) \leqslant \delta^{-C}$ by part (a). Moreover, by our choice of δ, we have that $\delta q \geqslant 1$, hence we conclude that $(1-\delta)^q\binom{N}{2} < 1$. So by part (b), we conclude that $r(H;q) > N$, which implies the claimed lower bound. □

Homework 11

Exercise 1 There is a natural analogue of the stepping-up construction going from uniformity 2 to 3. Namely, given a coloring $\chi : E(K_M^{(2)}) \to \{\text{red}, \text{blue}\}$, we can define $\psi : E(K_N^{(2)}) \to \{\text{red}, \text{blue}\}$, where $N = 2^M$, by

$$\psi(\{x, y, z\}) \coloneqq \chi(\delta(x, y), \delta(y, z)),$$

for $x < y < z$. Prove that this construction does not work, in the following sense: if χ contains a monochromatic $K_k^{(2)}$, then ψ contains a monochromatic $K_{2^k}^{(3)}$. Conclude that this contstruction cannot prove a better lower bound than $r_3(m) \geqslant 2^{\Omega(m)}$.

Solution. Suppose there is a monochromatic $K_k^{(2)}$ in χ, and suppose for simplicity that the vertices of this $K_k^{(2)}$ are $M-k+1, \ldots, M$, where we identify $V(K_M)$ with $\llbracket M \rrbracket$. Recall that we think of $V(K_N)$ as the leaves of a complete binary tree of depth M. Note that among the first 2^k leaves in this tree, that is among the vertices $v_1, \ldots, v_{2^k}$, we have that $\delta(v_i, v_j) \in \{M-k+1, \ldots, M\}$. Indeed, these first 2^k vertices are all descendants of a single node in the tree at depth $M-k+1$, hence the δ of any pair—the depth of the first common ancestor—is at least $M-k+1$. But by the definition of ψ, this shows that these 2^k vertices form a monochromatic $K_{2^k}^{(3)}$ under ψ.

In particular, if we want to obtain a lower bound on $r_3(m)$ in this way, we would need to set $m = 2^k + 1$ for some k. But then this construction would only show that

$$r_3(m) \geqslant 2^{r_2(k+1)-1}.$$

But since k is logarithmic in m, this cannot do better than proving an exponential lower bound on $r_3(m)$. □

Exercise 2

(b) Let S_k be the 3-uniform hypergraph with vertex set $w_0, \dots, w_k$ whose hyperedges are all triples $\{w_0, w_i, w_j\}$ for $1 \leqslant i, j \leqslant k$. By following the proof of Theorem 10.1.4, prove that

$$r_3(S_k, K_{k+1}^{(3)}) \leqslant 2^{Ck^4}$$

for some absolute constant $C > 0$.

(c) By coloring randomly, prove that

$$r_3(S_k, K_{k+1}^{(3)}) \geqslant 2^{ak^c},$$

for some absolute constants $a, c > 0$. What is the largest value of c you can obtain?

Solution.

(b) Let $N = 2^{Ck^4}$ for an appropriate constant C, and fix a 2-coloring of $E(K_N^{(3)})$. As in the proof of Theorem 10.1.4, we begin by constructing a sequence of vertices $w_1, \dots, w_t, w_{t+1}$ with the property that there is a coloring $\chi : E(K_t^{(2)})$ such that

$$\psi(\{w_i, w_j, w_\ell\}) = \chi(\{w_i, w_j\})$$

for all $1 \leqslant i < j < \ell \leqslant t + 1$. As in the proof of Theorem 10.1.4, we have to pay a factor of 2^i in order to find vertex i in this sequence. In other words, we can construct this sequence up to w_{t+1} so long as $N \geqslant 2^{1+\binom{t}{2}}$. By our choice of N, we conclude that we can do this up to $t = k^2$.

We now consider the coloring χ of $E(K_t^{(2)})$. By part (a), this coloring either contains a blue K_k or a red red *ordered* $K_{1,k-1}$ (where the central vertex precedes the $k-1$ leaves in the ordering). In the first case, as in the proof of Theorem 10.1.4, this blue K_k in χ yields a blue $K_{k+1}^{(3)}$ in ψ. In the second case, we claim that we obtain a red copy of S_k. Indeed, say for simplicity that $w_1, \dots, w_k$ form the ordered red copy of $K_{1,k-1}$ under χ. Then every hyperedge of the form $\{w_1, w_i, w_j\}$, where $2 \leqslant i < j \leqslant k$ is red under ψ. Moreover, we can also add w_{t+1} to this, and obtain the desired red copy of S_k.

(c) Let $N = k^{k/18}$ and $p = \frac{1}{\sqrt{k}} = N^{-9/k}$. We color every edge of $K_N^{(3)}$ randomly, giving it color red with probability p and blue with probability $1-p$, and making these choices independently.

First, we estimate the probability that there is a red copy of S_k. For a given set of $k+1$ vertices, the probability that it forms a copy of S_k is at most $(k+1)p^{\binom{k}{2}}$, where we have $k+1$ choices for which vertex is w_0, and then we need $\binom{k}{2}$ edges to all be colored red. By the union bound, the probability that some set of $k+1$ vertices form a red S_k is at most

$$\binom{N}{k+1} \cdot (k+1) \cdot p^{\binom{k}{2}} \leqslant N^{2k} p^{k^2/3} = (N^2 p^{k/3})^k = (N^2 N^{-3})^k < \frac{1}{2},$$

where we plug in our choice of $p = N^{-9/k}$. On the other hand, the probability that there is a blue $K_{k+1}^{(3)}$ is similarly at most

$$\binom{N}{k+1}(1-p)^{\binom{k+1}{3}} \leqslant N^{2k} e^{-pk^3/6} = (N^2 e^{-pk^2/6})^k = \exp\left(2 \ln N - \frac{pk^2}{6}\right)^k.$$

Now, we have that

$$2 \ln N - \frac{pk^2}{6} = \frac{k}{9} \ln k - \frac{k^{3/2}}{6},$$

which is a quantity tending to $-\infty$ as $k \to \infty$. Hence, for sufficiently large k, we have that this quantity is less than -1, and hence the probability of having a blue $K_{k+1}^{(3)}$ is also less than $\frac{1}{2}$. This shows that $r_3(S_k, K_{k+1}^{(3)}) > N$ for k sufficiently large. This in particular implies the claimed bound, with $c = 1$. □

Exercise 3(b) A *hyperforest* is a t-uniform hypergraph $\mathcal{H}$ with the following property. The hyperedges of $\mathcal{H}$ may be ordered as $e_1, \ldots, e_m$ so that, for every $2 \leqslant i \leqslant m$, we have $e_i \cap \bigcup_{j=1}^{i-1} e_j \subset e_{j'}$ for some $1 \leqslant j' \leqslant i-1$. In other words, each edge e_i is obtained as follows: we pick some $e_{j'}$, for $1 \leqslant j' \leqslant i-1$, pick some subset $S \subset e_{j'}$, and define e_i to consist of S plus $t - |S|$ new vertices, which were not yet used in any of $e_1, \ldots, e_{i-1}$.

Prove that for any $t, q \geqslant 2$, there exists some $C_{t,q} > 0$ such that the following holds. If $\mathcal{H}$ is a t-uniform hyperforest on n vertices, then

$$r_t(\mathcal{H}; q) \leqslant C_{t,q} n.$$

Hints: What are good analogues of Lemmas 5.2.2 and 5.2.3 in the t-uniform setting? Why do we require e_i to be glued along a subset of some $e_{j'}$, rather than on a subset of $\bigcup_{j=1}^{i-1} e_j$?

Solution. Let $\mathcal{G}$ be a t-uniform hypergraph on $N \geqslant n$ vertices and with at least tnN^{t-1} hyperedges. Let us call a set $S \subseteq V(\mathcal{G})$ *poor* if $|S| \leqslant t-1$ and if $|S|$ lies at least 1 but fewer than $n\binom{N}{t-1-|S|}$ hyperedges (that is, the number of vertices x such that $S \cup \{x\} \in E(\mathcal{G})$ is between 1 and $n\binom{N}{t-1-|S|} - 1$). We repeatedly do the following operation: if there is a poor set S, we delete from $E(\mathcal{G})$ all hyperedges containing S.

In each step of this process, we delete strictly fewer than $n\binom{N}{t-1-|S|}$ hyperedges, by the definition of a poor set. Moreover, if we delete all edges containing S, we will never do so again, since at this point S will never again be poor (since it is now contained in zero hyperedges). Sicne there are $\binom{N}{s}$ sets S of size s, in this process, the total number of edges we delete from $\mathcal{G}$ is less than

$$\sum_{s=0}^{t-1} n\binom{N}{t-1-s}\binom{N}{s} < \sum_{s=0}^{t-1} nN^{t-1-s} \cdot N^s = tnN^{t-1}$$

In other words, we have proved the following lemma: if $|E(\mathcal{G})| \geqslant tnN^{t-1}$, then there is a subhypergraph $\mathcal{G}' \subseteq \mathcal{G}$ with $V(\mathcal{G}') = V(\mathcal{G})$ and $E(\mathcal{G}') \neq \varnothing$, such that no set is poor in $\mathcal{G}'$.

Now, let $\mathcal{H}$ be any t-uniform hyperforest with n vertices. We claim that any $\mathcal{G}'$ with the property above contains $\mathcal{H}$ as a subhypergraph. This is proved in essentially the same way as Lemma 5.2.3, namely by induction on m, the number of edges of $\mathcal{H}$. The base case $m = 1$ is trivial, since we assumed that $E(\mathcal{G}') \neq \varnothing$ and that $\mathcal{G}'$ has $N \geqslant n$ vertices. Inductively, suppose we have proved the claim for $m-1$, and let $\mathcal{H}$ have m edges. Order these edges as $e_1, \ldots, e_m$, as in the definition of a hyperforest. Let $\mathcal{H}'$ be obtained from $\mathcal{H}$ by deleting e_m and all vertices which are used in no hyperedge other than e_m. $\mathcal{H}'$ is another hyperforest, now with $m-1$ hyperedges, so by the inductive hypothesis we have $\mathcal{H}' \subseteq \mathcal{G}'$. By the definition of a hyperforest, we have that $e_m \cap V(\mathcal{H}') \subseteq e_{j'}$ for some $j' \leqslant m-1$, that is, that e_m contains some set $S \subseteq e_{j'}$, plus $t - |S|$ new vertices not appearing in $V(\mathcal{H}')$. Consider how $e_{j'}$ is embedded into $\mathcal{G}'$. The set S is contained in $e_{j'} \in E(\mathcal{G}')$, hence it is contained in at least one hyperedge of $\mathcal{G}'$, implying that it is contained in at least $n\binom{N}{t-1-|S|}$ such hyperedges, as it is not poor. However, the total number of these hyperedges that use at least one vertex already used in embedding $V(\mathcal{H}') \setminus S$ is at most $(n-1)\binom{N}{t-1-|S|}$, since we have $|V(\mathcal{H}') \setminus S| \leqslant n-1$ choices for this repeated vertex, and then at most $\binom{N}{t-1-|S|}$ choices for the remaining vertices. In other words, we are able to find a hyperedge of $\mathcal{G}'$ containing S and which uses no vertices of $V(\mathcal{H}') \setminus S$. Then we may extend the embedding of $\mathcal{H}'$ to an embedding of $\mathcal{H}$ by sending e_m to this new hyperedge.

What we have proved, therefore, is that if $V(\mathcal{G}) = N \geqslant n$ and $|E(\mathcal{G})| \geqslant tnN^{t-1}$, then $\mathcal{G}$ contains every n-vertex hyperforest as a subhypergraph (because it contains a non-empty subhypergraph $\mathcal{G}'$ with no poor sets, which in turn contains $\mathcal{H}$). Finally, let $C_{t,q}$ be sufficiently large so that

$$\frac{1}{q}\binom{C_{t,q}n}{t} \geqslant tn(C_{t,q}n)^{t-1}$$

for all $n \geqslant t$. Note that such a $C_{t,q}$ exists, and depends only on t, q, since the main term of the left-hand side is $C_{t,q}^t n^t/(t!q)$, whereas the main term of the right-hand side is $C_{t,q}^{t-1} tn^t$, which is of lower order in $C_{t,q}$. Then let $N = C_{t,q}n$, and consider any q-coloring of $E(K_N^{(t)})$. One of the color classes must have at least $\frac{1}{q}\binom{N}{t}$ edges, which by the above implies that this color contains any n-vertex hyperforest as a subgraph. $\square$

Homework 12

Exercise 2 Let $N = r_3(k)$, and let $p_1, \ldots, p_N$ be points in $\mathbb{R}^2$ with no three collinear. Define $\chi : E(K_N^{(3)}) \to \{\text{even}, \text{odd}\}$ by

$$\chi(\{i, j, \ell\}) \coloneqq \begin{cases} \text{even} & \text{if there are an even number of points } p_m \text{ in the triangle } p_i p_j p_\ell, \\ \text{odd} & \text{otherwise.} \end{cases}$$

Prove that a monochromatic $K_k^{(3)}$ under χ corresponds to k points in convex position. Conclude that $\mathrm{Kl}(k) \leqslant r_3(k)$, and in particular obtain a new proof of Theorem 10.3.4.

Solution. Let p_1, p_2, p_3, p_4 be four points not in convex position, say p_4 is contained in the convex hull of p_1, p_2, p_3. Every point p_m in the convex hull of p_1, p_2, p_3, apart from p_4 itself, must lie in exactly one of the three triangles $\{p_1, p_2, p_4\}, \{p_1, p_3, p_4\}, \{p_2, p_3, p_4\}$. That is, the total number of points in the triangle $\{p_1, p_2, p_3\}$ is the sum of the number of points in the three smaller triangles, plus one (for the point p_4). In particular, if all three smaller triangles contain an even number of points, then the large triangle contains an odd number of points, and vice versa.

Therefore, in any monochromatic $K_k^{(3)}$ under χ, every set of 4 points is in convex position. Therefore, every monochromatic $K_k^{(3)}$ under χ yields a set of k points in convex position, by Lemma 10.3.5. That shows that if $N = r_3(k)$, then any collection of N points in the plane, no three collinear, contains k in convex position. This proves $\mathrm{Kl}(k) \leqslant r_3(k)$. □

Exercise 3 A collection of points in $\mathbb{R}^d$ is said to be *in general position* if no $d+1$ of them lie on a $(d-1)$-dimensional hyperplane. (So in two dimensions, this says that no three are collinear, in three dimensions it says that no four are coplanar, etc.)

(a) Prove that among any $d+3$ points in $\mathbb{R}^d$ which are in general position, there are $d+2$ in convex position.

(b) Given $k \geqslant d+2$, let $N = r_{d+2}(d+3, k)$. Prove that among any N points in $\mathbb{R}^d$ in general position, there are k in convex position.

(c) Prove that among any $\mathrm{Kl}(k)$ points in $\mathbb{R}^d$, no three collinear, there are k in convex position.

[This is stronger than the result in (b) in two ways: the bound is *independent* of d, and the assumption is weakened from general position to no three collinear.]

Solution.

(a) Fix points $p_1, \ldots, p_{d+3}$ in $\mathbb{R}^d$ which are in general position. Let P be their convex hull; P is a convex polytope in $\mathbb{R}^d$, so it has at least $d+1$ vertices. If P has at least $d+2$ vertices, then these vertices form a collection of at least $d+2$ points in convex position, and we are done. Therefore, we may assume that P has exactly $d+1$ vertices, say $p_1, \ldots, p_{d+1}$, and that the remaining two points p_{d+2}, p_{d+3} are in the interior of P.

Consider the hyperplane H passing through the points $p_1, \ldots, p_{d-2}, p_{d+2}, p_{d+3}$ (this is a collection of d points in general positions, so there is a unique hyperplane passing through them). By the general position assumption, the three remaining points p_{d-1}, p_d, p_{d+1} do not lie on H. Therefore, at least two of them must lie on the same side of H. Say without loss of generality that p_d and p_{d+1} lie on the same side of H.

We now claim that $p_1, \ldots, p_{d-2}, p_{d+2}, p_{d+3}, p_d, p_{d+1}$ (i.e. all points except p_{d-1}) are in convex position. Geometrically, this should hopefully be clear (the picture in any dimension is essentially the same as the 2-dimensional picture, where we encountered the same fact in the proof of Proposition 10.3.3). For a formal proof, it suffices to show that each of these points is a vertex of their convex hull. For the original vertices of P, that is, all points except p_{d+2}, p_{d+3}, this is immediate—the vertices of P are not in the convex hull of all of the remaining points, hence they are also not in the convex hull of any subset of the remaining points. So we only need to verify that p_{d+2}, p_{d+3} are vertices.

By translating rotating $\mathbb{R}^d$, we may assume that H is the hyperplane defined by the equation $x_d = 0$, and that p_d, p_{d+1}, which lie on one side of H, both have final coordinate that is positive. But this shows that they cannot be used in any convex combination yielding p_{d+2} or p_{d+3}, both of which have final coordinate equal to 0. So it suffices to prove that p_{d+2}, p_{d+3} are vertices of the convex hull of the points in H. But this is clear, since H is a copy of $\mathbb{R}^{d-1}$, and we have put on it d points in general position, which must therefore be in convex position.

(b) Let $N = r_d(d+3, k)$, and let $p_1, \ldots, p_N$ be points in general position in $\mathbb{R}^d$. Define a coloring $\chi : E(K_N^{(d+2)}) \to \{\text{red}, \text{blue}\}$ by coloring a $(d+2)$-tuple red if it is in convex position, and blue otherwise. By part (a), there cannot be a red $K_{d+3}^{(d+2)}$ in this coloring, so there must exist a blue $K_k^{(d+2)}$. But by the d-dimensional generalization of Lemma 10.3.5, this is a set of k points in convex position.

(c) Let $N = \mathrm{Kl}(k)$, and let $p_1, \ldots, p_N$ be N points in $\mathbb{R}^d$ with no three collinear. Pick a random 2-dimensional plane H through the origin in $\mathbb{R}^d$ and let $q_1, \ldots, q_N$ be the orthogonal projections of $p_1, \ldots, p_N$ onto H. Note that for every triple i, j, ℓ, it is a measure-zero event that q_i, q_j, q_ℓ are collinear (this event is precisely the event that the plane containing these points is orthogonal to H). Since there are finitely many such triples, we see that with probability 1, the points $q_1, \ldots, q_n \in H \cong \mathbb{R}^2$ have no collinear triples. By Theorem 10.3.4, there exist k of these points in convex position, say $q_1, \ldots, q_k$. But this implies that $p_1, \ldots, p_k$ are also in convex position, since if, say, p_k were in the convex hull of $p_1, \ldots, p_{k-1}$, then also q_k would be in the convex hull of $q_1, \ldots, q_{k-1}$. $\square$

Exercise 5 Prove that any sequence of (not necessarily distinct) real numbers of length $(k-1)^3+1$ contains a subsequence of length k that is strictly increasing, strictly decreasing, or constant. Prove that this bound is best possible.

Solution. Let $N = (k-1)^3 + 1$, and let $a_1, \ldots, a_N$ be a sequence of real numbers. Let $\delta(m), \iota(m), \kappa(m)$ be the length of the longest strictly decreasing, strictly increasing, and constant sequence, respectively, ending at a_m. If $\delta(m) \geqslant k, \iota(m) \geqslant k$, or $\kappa(m) \geqslant k$ for any m, we are done, so we may assume that $1 \leqslant \delta(m), \iota(m), \kappa(m) \leqslant k-1$ for all m. So there are only $(k-1)^3$ options for the triple $(\delta(m), \iota(m), \kappa(m))$, so there must exist two indices $\ell < m$ such that $\delta(\ell) = \delta(m), \iota(\ell) = \iota(m), \kappa(\ell) = \kappa(m)$. But this is impossible: if $a_\ell < a_m$ then $\iota(\ell) < \iota(m)$, if $a_\ell > a_m$ then $\delta(\ell) < \delta(m)$, and if $a_\ell = a_m$ then $\kappa(\ell) < \kappa(m)$. This contradiction completes the proof.

To see that this bound is tight, we first consider the following sequence of numbers:

$$k-1, k-2, \ldots, 2, 1, \quad 2(k-1), 2(k-1)-1, \ldots, k+1, k, \quad \ldots \quad (k-1)^2, (k-1)^2-1, \ldots, (k-1)^2-(k-2).$$

In other words, we have $k-1$ decreasing sequences of length $k-1$, where these decreasing sequences are arranged in increasing order. This is a sequence of $(k-1)^2$ distinct real numbers. Note that the longest decreasing subsequence has length $k-1$, since no sequence intersecting two "blocks" can be decreasing. Moreover, the longest increasing subsequence also has length $k-1$, since no sequence using two elements from a single block can be increasing.

Now, simply repeat each element of this sequence $k-1$ times. We thus obtain a sequence of $(k-1)^3$ real numbers; we have not increased the length of the longest strictly increasing or strictly decreasing subsequence, and of course every constant subsequence has length at most $k-1$. This shows that the result above is best possible, in that it becomes false if we replace $(k-1)^3+1$ by $(k-1)^3$. □

Exercise 6 Let us say that a coloring of $E(K_k)$ is *semi-starry* if the vertices can be sorted as $v_1, \ldots, v_k$ such that all edges $v_i v_j$, where $j > i$, are of the same color. (The only difference from a starry coloring is that we do not require these colors to be distinct.)

(a) Prove that if $N \geqslant (k-1)^2 + 1$, then any semi-starry coloring of $E(K_N)$ contains a monochromatic or starry K_k. Such a result was implicitly used in the proof of Theorem 11.2.2.

(b) Prove that if $N \geqslant k^{4k}$, then any coloring of $E(K_N)$, with an arbitrary number of colors, contains a rainbow or a semi-starry K_k.

(c) Show that there exists a coloring of $E(K_N)$, where $N = k^k$, with no rainbow or semi-starry K_k. Thus, the result of part (b) is best possible up to the constant factor in the exponent.

Solution.

(a) Let $N \geqslant (k-1)^2+1$, and let $E(K_N)$ be colored in a semi-starry way. Let the vertices of K_N be $v_1, \dots, v_N$, sorted according to this semi-starry coloring. Let $c_1, \dots, c_N$ be the colors used in this coloring, i.e. all edges v_iv_j where $j > i$ receive color c_i. If k of these colors are equal, then the corresponding vertices form a monochromatic K_k. If not, then every color is used at most $k-1$ times, meaning that at least k distinct colors are used; any k vertices using distinct colors form a starry K_k.

(b) We argue as in the proof of Theorem 11.2.2. Let $N = k^{4k}$, and fix an arbitrary coloring of $E(K_N)$. We let $S_0 = V(K_N)$. We now run the following process, for all $i \geqslant 1$.

  (a) If $|S_{i-1}| < 2$, stop the process.

  (b) If every vertex in S_{i-1} is incident to at most $|S_{i-1}|/k^4$ edges in each color, we apply Lemma 11.2.3 to S_{i-1} with $M = |S_{i-1}| \geqslant 2$. We conclude that S_{i-1} contains a rainbow K_k, completing the proof.

  (c) If not, there is some vertex $v_i \in S_{i-1}$ and some color c_i such that v_i is incident to at least $|S_{i-1}|/k^4$ edges of color c_i in S_{i-1}. We let S_i be the c_i-colored neighborhood of v_i in S_{i-1}.

  (d) Increment i by 1 and return to step (a).

If we ever find a rainbow K_k in this process, we are done, so we may assume that that never happens. Note that as long as the process continues, we have that $|S_i| \geqslant |S_{i-1}|/k^4$, so by induction we have that $|S_i| \geqslant k^{4(k-i)}$. Hence we can continue this process at least until step $i-1 = k-1$. In other words, this process produces a sequence $v_1, \dots, v_k$ of vertices and $c_1, \dots, c_k$ of colors, with the property that each v_i is adjacent in color c_i to all v_j with $j > i$. But this is precisely a semi-starry K_k, so we are again done.

(c) By coloring randomly with $\binom{k}{2} - 1$ colors, we have no rainbow K_k, and one can show that there is no semi-starry K_k with positive probability as long as $N = k^{k-o(k)}$. This is almost the claimed bound, but to get the precise result claimed we use a deterministic construction.

Let $\ell = \binom{k}{2} - 1$. Divide $V(K_N)$ into ℓ parts $A_1, \dots, A_\ell$. Define colors $c_0, \dots, c_{\ell-1}$, and color all edges between A_i and A_j with color $c_{(i+j) \bmod \ell}$. Note that this implies that each block A_i is joined by a distinct color to each other block.

We now iterate this construction inside each A_i. Namely, we divide each A_i into $A_{i,1}, \dots, A_{i,\ell}$, and color all edges between $A_{i,j}$ and $A_{i,j'}$ by color $c^{(1)}_{(j+j') \bmod \ell}$, where the colors $c^{(1)}_0, \dots, c^{(1)}_{\ell-1}$ are a set of ℓ *new* colors. We keep iterating this down to $k-1$ layers. Thus, we can do this construction for $N = \ell^{k-1}$.

We claim that this coloring has no rainbow or semi-starry K_k. To see this, consider k vertices in $V(K_N)$. If they all lie in distinct parts A_i, then they do not form a semi-starry K_k (since each vertex would be incident to $k-1$ edges of different colors), and they do not form a rainbow K_k since only $\ell < \binom{k}{2}$ colors are used between the parts

$A_1, \ldots, A_\ell$. Therefore, at least two vertices must be in the same part A_i. We now claim that in fact, at least $k-1$ of the vertices have to be in the same part. Indeed, if this is not the case, then there are either two parts, say A_1, A_2, each containing at least two vertices, or there are three parts, say A_1, A_2, A_3 such that there are two vertices in A_1 and at least one vertex in each of A_2, A_3. In the first case, we do not have a rainbow K_k, since all four edges between A_1 and A_2 receive the same color, and we do not have a semi-starry K_k since the edges inside A_1, A_2 receive distinct colors from those between A_1, A_2. Similarly, in the second case, we do not have a rainbow or a semi-starry K_k.

Therefore, it must be the case that at least $k-1$ of the vertices lie in the same part A_i. But we may now repeat this argument in the second layer of the construction, and conclude that at least $k-2$ vertices lie in the same part $A_{i,j}$. Continuing all the way down, and recalling we only do $k-1$ layers of iteration, we see that there is no rainbow or semi-starry K_k in this coloring.

The final thing is to note that

$$N = \ell^{k-1} = \left(\binom{k}{2} - 1\right)^{k-1} \geqslant k^k,$$

where the final inequality holds for all $k \geqslant 5$. $\square$

Exercise 7 Prove the bipartite canonical Ramsey theorem, which states the following. For every $k \geqslant 2$, there exists some N such that in any coloring of $E(K_{N,N})$, with an arbitrary number of colors, there is a $K_{k,k}$ which is monochromatic, rainbow, or starry.

(Here, a $K_{k,k}$ is *rainbow* if all k^2 edges receive different colors, and is *starry* if it is colored by exactly k distinct colors, each of whose color classes is a star $K_{1,k}$.)

Solution. We begin by proving the following lemma, a bipartite analogue of Lemma 11.2.3. Consider a complete bipartite graph with parts X, Y, and suppose that its edges are colored with an arbitrary number of colors. If every $x \in X$ is incident to at most $|Y|/k^5$ edges in every color, and every $y \in Y$ is incident to at most $|X|/k^5$ edges in every color, then there is a rainbow $K_{k,k}$ in this coloring.

To prove this, we let $x_1, \ldots, x_k \in X, y_1, \ldots, y_k \in Y$ be uniformly random sets of k distinct vertices in each part (that is, the vertices in X are chosen uniformly at random from all $\binom{|X|}{k}$ options, and similarly for Y). We claim that with positive probability they form a rainbow $K_{k,k}$. To see this, let us first estimate the probability that $x_i y_j$ and $x_i y_\ell$ receive the same color, for indices $i, j, \ell \in [\![k]\!]$. Conditioning on the outcome of x_i, y_j, there are at most $|Y|/k^5$ vertices $y \in Y$ such that $x_i y$ has the same color as $x_i y_j$, and y_ℓ is chosen uniformly at random from $Y \setminus \{y\}$. Hence the conditional probability that $\chi(x_i y_\ell) = \chi(x_i y_j)$ is at most

$$\frac{1}{|Y|-1} \cdot \frac{|Y|}{k^5} < \frac{2}{k^5}.$$

Since this upper bound holds regardless of the outcome of the conditioning, we conclude that it holds without conditioning. There are at most $k^3/2$ choices for i, j, ℓ (since this event is unchanged if we swap j and ℓ), so by the union bound the probability that $\chi(x_i y_j) = \chi(x_i y_\ell)$ for some such triple is at most $(k^3/2) \cdot (2/k^5) = 1/k^2 \leqslant \frac{1}{4}$, since $k \geqslant 2$. By the exact same argument, the probability that $\chi(x_j y_i) = \chi(x_\ell y_i)$ for some such triple is at most $\frac{1}{4}$.

Similarly, we find that the probability that $\chi(x_i y_j) = \chi(x_\ell y_m)$ is at most $2/k^5$. Since there are at most $k^4/4$ choices for these indices (since we may swap i, ℓ and j, m), the probability that this happens for any such 4-tuple is at most $(k^4/4) \cdot (2/k^4) = 1/(2k) \leqslant \frac{1}{4}$. So the probability that our chosen $K_{k,k}$ is *not* rainbow is at most $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} < 1$, hence there must exist a rainbow $K_{k,k}$ in the coloring. This proves the lemma.

We now let $N = k^{10k^2}$, and argue as follows. Let X_0, Y_0 be sets of size N, and color the complete bipartite graph between X_0 and Y_0 with an arbitrary number of colors. We repeatedly do the following algorithm: if there is some $x_i \in X_i$ with at least $|Y_i|/k^5$ neighbors in some color c_i^X, we set $X_{i+1} = X_i \setminus \{x_i\}$ and set Y_i to be the c_i^X-colored neighborhood of x_i in Y_i. Similarly, if there is some $y_i \in Y_i$ with at least $|X_i|/k^5$ neighbors in some color c_i^Y, we set $Y_{i+1} = Y_i \setminus \{y_i\}$ and X_{i+1} to be the c_i^Y-colored neighborhood of y_i in X_i. If neither of these options is possible, then we apply the lemma above to find a rainbow $K_{k,k}$, so we may assume that we can always keep this process going. Note that at each step, we have $|X_{i+1}| \geqslant |X_i|/k^5, |Y_{i+1}| \geqslant |Y_i|/k^5$, since at every step we either remove one vertex from X_i (resp. Y_i) or shrink it by a factor of k^5. Hence, by the choice of N, we can keep this process going until step $i = 2k^2$. By the pigeonhole principle, at least half of the steps must have been done from one side, say X, so we pulled out k^2 special vertices from X, associated with k^2 special colors. If k of these colors are identical we find a monochromatic $K_{k,k}$, and if not then we must have used at least k distinct colors, yielding a starry $K_{k,k}$. $\square$