1. (a) Prove that for any positive integer q, there exists a positive integer $N = N(q)$ such that the following holds. For any q-coloring of $[\![N]\!]$, there exist $x, y, z \in [\![N]\!]$ such that $x, y, z, x+y, y+z, x+y+z$ all receive the same color. (Note that $x+z$ is omitted!)

   (b) Generalize the previous part as follows. Prove that for all positive integers q, t, there exists a positive integer $N = N(q,t)$ such that the following holds. For any q-coloring of $[\![N]\!]$, there exist $x_1, \ldots, x_t \in [\![N]\!]$ such that the sums $\sum_{i=a}^{b} x_i$ all receive the same color, for all non-empty $1 \leqslant a \leqslant b \leqslant t$.

   (c) Prove that in part (b), one can moreover ensure that the numbers $x_1, \ldots, x_t$ are all distinct.

✣ 2. In class, we proved that $r(k) < 4^k$ using the Erdős–Szekeres argument. Ramsey's original proof used a *different* argument, which yielded the worse bound $r(k) \leqslant k!$. Find a natural argument yielding this bound. (That is, don't simply quote or rederive the Erdős–Szekeres argument!)

3. (a) Prove that $r(3,3) = 6, r(3,4) = 9$, and $r(4,4) \leqslant 18$.

   ⋆ (b) Prove that $r(4,4) = 18$.

   ? (c) The best known bounds on $r(5,5)$ are $43 \leqslant r(5,5) \leqslant 48$. Can you improve either of these bounds?

4. (a) By more carefully analyzing the proof of Theorem 2.2.2 in the notes, prove that

$$r(k) > \left(\frac{1}{e\sqrt{2}} - o(1)\right) k2^{k/2}.$$

   ⋆ (b) Improve this bound by a constant factor.

   ? (c) Improve this bound by a super-constant factor.

5. Given two graphs G, H, their *lexicographic product* $G \cdot H$ is defined as follows. Its vertex set is $V(G \cdot H) = V(G) \times V(H)$, and two vertices $(a,b), (c,d)$ are adjacent if either $ac \in E(G)$ or $a = c$ and $bd \in E(H)$.

   (a) Compute the size of the largest clique and the largest independent set in $G \cdot H$.

   (b) Prove that the Ramsey number $r(k)$ satisfies $r(k+1) > k^{\log_2(5)}$ for all k that are powers of 2.

   [Note that this already disproves Turán's belief that $r(k)$ may grow only quadratically as a function of k.]

⋆ means that a problem is hard.

? means that a problem is open.

✣ means that a problem is not directly related to the topic of the course.

$\star$ (c) Using the same approach, find an *explicit* construction of a coloring witnessing that $r(k)$ grows super-polynomially in k. In other words, for any $C > 0$ and any sufficiently large k, find an explicit 2-coloring of $E(K_N)$, where $N = k^C$, with no monochromatic clique of order k.

$\star$ (d) By carefully working through the dependencies in (c), prove via an explicit coloring that

$$r(k) > k^{c \frac{\log \log \log k}{\log \log \log \log k}}$$

for some absolute constant $c > 0$. Can you further improve this bound?

? (e) Can you use such an approach to resolve Open problem 2.2.3?