1. A function $\varphi : [\![k]\!]^s \to [\![k]\!]^d$ is the same as a tuple $\varphi = (\varphi_1, \ldots, \varphi_d)$ of functions $\varphi_j : [\![k]\!]^s \to [\![k]\!]$. Such a function φ is called a *combinatorial mapping* if every component φ_j is either a constant function or a coordinate function, i.e. $\varphi_j(x_1, \ldots, x_s) = x_i$ for some i. An *s-dimensional combinatorial subspace* of $[\![k]\!]^d$ is the image of a combinatorial mapping $\varphi : [\![k]\!]^s \to [\![k]\!]^d$ which is furthermore injective.

   (a) Prove that a 1-dimensional combinatorial subspace is the same as a combinatorial line, and convince yourself that this is a reasonable generalization of combinatorial lines for $s \geqslant 2$.

   (b) Show that s-dimensional combinatorial subspaces of $[\![k]\!]^d$ are in bijection with *s-roots*, which are words $\rho \in \{1, \ldots, k, *_1, \ldots, *_s\}^d$ in which each star symbol $*_i$ appears at least once.

   (c) Prove that for every $k, s, q \geqslant 1$, there exists some d such that any q-coloring of $[\![k]\!]^d$ contains a monochromatic s-dimensional combinatorial subspace.

   Hint: Prove that $d = s \cdot \mathrm{HJ}(k^s; q)$ suffices.

2. (a) Suppose that there is a coloring $\chi : [\![N]\!]^t \to [\![q]\!]$ with no homothetic copy of

   $$S \coloneqq \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}.$$

   Using χ, construct a protocol for t players to compute the exactly-N function using at most $t\lceil \log q \rceil$ bits of communication in the number-on-the-forehead model.

   (b) Reinterpret the result of (a) as saying the following: If the Gallai–Witt theorem is false for this choice of S, then there *is* a protocol to compute the exactly-N function using only a constant number of bits of communication.

   In other words, we proved in Theorem 9.4.1 that the Gallai–Witt theorem implies a super-constant lower bound for this communication complexity, and (a) gives a converse: a super-constant lower bound for this communication complexity implies the Gallai–Witt theorem for this choice of S.

   (c) Improve your protocol in (a) to one using only $t + \lceil \log q \rceil$ bits of communication.

3. Prove the density Hales–Jewett theorem for $k = 2$. In other words, prove that for every $\delta > 0$ and every sufficiently large d, every subset $A \subseteq [\![2]\!]^d$ with $|A| \geqslant \delta 2^d$ contains a combinatorial line.

4. Prove that there is no density version of Schur's theorem.

5. Let us say that a graph H has the *density Ramsey property* if for every $\delta > 0$ and every sufficiently large N, any N-vertex graph G with at least $\delta\binom{N}{2}$ edges has a copy of H.

   (a) Show that if H has the density Ramsey property, then $r(H; q)$ is finite for all q, by applying the definition with $\delta = \frac{1}{q}$.

   [This exercise is of course a bit silly, since we already know that $r(H; q)$ is finite—the point is just to understand how such density results are stronger than the corresponding coloring results, just as Szemerédi's theorem is stronger than van der Waerden's theorem.]

(b) Prove that if H is bipartite, then H has the density Ramsey property.

(c) Prove that if H is not bipartite, then H does not have the density Ramsey property.

6. The *finite unions theorem* states the following. For every $m, q \geqslant 2$, there exists some N such that in any q-coloring of $2^{[\![N]\!]}$ (that is, every subset of $[\![N]\!]$ receives some color), there exist disjoint sets $S_1, \ldots, S_m \subseteq [\![N]\!]$ such that all of the unions $\bigcup_{i \in I} S_i$, for $\varnothing \neq I \subseteq [\![m]\!]$, receive the same color.

   (a) Prove that the finite unions theorem implies Theorem 9.3.1.

   ⋆(b) Prove the finite unions theorem.

7. For a bipartite graph H and a number $\delta > 0$, let $r_d(H; \delta)$ denote the minimum integer N such that every N-vertex graph with at least $\delta \binom{N}{2}$ edges has a copy of H. (Note that this is a well-defined quantity, by problem 5(b).)

   (a) By examining your solution to problem 5(b), show that for every bipartite graph H, there exists some $C > 0$ such that

$$r_d(H; \delta) \leqslant \left(\frac{1}{\delta}\right)^C$$

   for all $0 < \delta \leqslant \frac{1}{2}$.

   (b) Let H be a graph, and suppose G is an N-vertex graph with $\delta \binom{N}{2}$ edges and with no copy of H. Prove that if q is an integer satisfying $(1 - \delta)^q \binom{N}{2} < 1$, then

$$r(H; q) > N.$$

   Hint: Randomly permute the vertices of G to obtain q copies $G_1, \ldots, G_q$. Show that with positive probability, every edge of K_N appears in at least one G_i.

   (c) Fix a bipartite graph H, and let C be the constant from part (a). Show that

$$r_d\left(H; \frac{2C \ln q}{q}\right) \leqslant r(H; q) \leqslant r_d\left(H; \frac{1}{q}\right),$$

   where the lower bound uses part (b) and the upper bound uses your solution to problem 5(a). This shows that $r(H; q)$ and $r_d(H; 1/q)$ are closely related for bipartite H.

   ⋆(d) Let $\mathrm{Sz}(k; \delta)$ denote the least N such that every $A \subseteq [\![N]\!]$ with $|A| \geqslant \delta N$ contains a k-AP. Using similar arguments, try to relate $W(k; q)$ to $\mathrm{Sz}(k; \delta)$, proving both upper and lower bounds involving $\delta \approx 1/q$.

   Hint: It may be helpful to work in $\mathbb{Z}/N$ rather than in $[\![N]\!]$.