1. There is a natural analogue of the stepping-up construction going from uniformity 2 to 3. Namely, given a coloring $\chi : E(K_M^{(2)}) \to \{\text{red}, \text{blue}\}$, we can define $\psi : E(K_N^{(2)}) \to \{\text{red}, \text{blue}\}$, where $N = 2^M$, by
$$\psi(\{x, y, z\}) \coloneqq \chi(\delta(x, y), \delta(y, z)),$$
for $x < y < z$. Prove that this construction does not work, in the following sense: if χ contains a monochromatic $K_k^{(2)}$, then ψ contains a monochromatic $K_{2^k}^{(3)}$. Conclude that this contstruction cannot prove a better lower bound than $r_3(m) \geqslant 2^{\Omega(m)}$.

2. (a) Let $V(K_M) = \{v_1, \ldots, v_M\}$. Prove that if $M \geqslant (k-1)^2+1$, then in every coloring $E(K_M) \to \{\text{red}, \text{blue}\}$, there is a blue K_k or a vertex v_i with at least $k-1$ red neighbors v_j, where $j > i$.
[Note that this is a stronger statement than saying $r(K_{1,k-1}, K_k) \leqslant (k-1)^2 + 1$, since in the second case we are finding an *ordered* copy of $K_{1,k-1}$.]
   (b) Let S_k be the 3-uniform hypergraph with vertex set $w_0, \ldots, w_k$ whose hyperedges are all triples $\{w_0, w_i, w_j\}$ for $1 \leqslant i, j \leqslant k$. By following the proof of Theorem 10.1.4, prove that
$$r_3(S_k, K_{k+1}^{(3)}) \leqslant 2^{Ck^4}$$
for some absolute constant $C > 0$.
   (c) By coloring randomly, prove that
$$r_3(S_k, K_{k+1}^{(3)}) \geqslant 2^{ak^c},$$
for some absolute constants $a, c > 0$. What is the largest value of c you can obtain?

3. A *hyperforest* is a t-uniform hypergraph $\mathcal{H}$ with the following property. The hyperedges of h may be ordered as $e_1, \ldots, e_m$ so that, for every $2 \leqslant i \leqslant m$, we have $e_i \cap \bigcup_{j=1}^{i-1} e_j \subset e_{j'}$ for some $1 \leqslant j' \leqslant i-1$. In other words, each edge e_i is obtained as follows: we pick some $e_{j'}$, for $1 \leqslant j' \leqslant i-1$, pick some subset $S \subset e_{j'}$, and define e_i to consist of S plus $t - |S|$ new vertices, which were not yet used in any of $e_1, \ldots, e_{i-1}$.
   (a) Prove that in case $t = 2$, a graph is a 2-uniform hyperforest if and only if it is a forest in the usual sense, i.e. a graph with no cycles.
   (b) Prove that for any $t, q \geqslant 2$, there exists some $C_{t,q} > 0$ such that the following holds. If $\mathcal{H}$ is a t-uniform hyperforest on n vertices, then
$$r_t(\mathcal{H}; q) \leqslant C_{t,q} n.$$
Hints: What are good analogues of Lemmas 5.2.2 and 5.2.3 in the t-uniform setting? Why do we require e_i to be glued along a subset of some $e_{j'}$, rather than on a subset of $\bigcup_{j=1}^{i-1} e_j$?

$\star$ means that a problem is hard.
? means that a problem is open.
✥ means that a problem is on a topic beyond the scope of the course.

⋆ 4. Let $\mathcal{H}$ be the 3-uniform hypergraph with vertex set $\{v_1, \ldots, v_k\}$ whose hyperedges are all triples $\{v_i, v_{i+1}, v_j\}$, where $i, j \in [\![k]\!]$ with $j \notin \{i, i+1\}$. Here, indices are taken modulo k, so that $\{v_k, v_1, v_j\}$ is a hyperedge of $\mathcal{H}$ for all $j \notin \{1, k\}$.

Prove that there exist constants $c, C > 0$ such that

$$2^{2^{ck}} \leqslant r_3(\mathcal{H}; 4) \leqslant 2^{2^{Ck}}.$$

In other words, $r_3(\mathcal{H}; 4)$ is roughly as large as $r(K_k^{(3)}; 4)$, even though $\mathcal{H}$ has $\Theta(k^2)$ edges while $K_k^{(3)}$ has $\Theta(k^3)$ edges.

[In the case of graphs, no such example can exist—any graph with substantially fewer edges than $K_k^{(2)}$ has a much smaller Ramsey number than $K_k^{(2)}$.]

⋆ 5. A 3-uniform hypergraph $\mathcal{H}$ is *tripartite* if its vertex set can be partitioned as $V_1 \cup V_2 \cup V_3$, so that every hyperedge of $\mathcal{H}$ uses exactly one vertex from each V_i. The *complete tripartite* 3*-uniform hypergraph* $K_{k,k,k}^{(3)}$ is the tripartite hypergraph where $|V_1| = |V_2| = |V_3| = k$ and in which every triple in $V_1 \times V_2 \times V_3$ is a hyperedge.

(a) Prove that there exist absolute constants $c, C > 0$ such that

$$2^{ck^2} \leqslant r_3(K_{k,k,k}^{(3)}) \leqslant 2^{Ck^2}.$$

⋆ (b) Let $\mathcal{H}$ be a tripartite 3-uniform hypergraph with $|V_1| = |V_2| = |V_3| = k$, and suppose that every vertex of $\mathcal{H}$ lies in at most Δ hyperedges. Prove that

$$r_3(\mathcal{H}) \leqslant k^{1+o(1)},$$

where the $o(1)$ term tends to 0 as $k \to \infty$, while keeping Δ constant.

⋆ (c) Extend these results to uniformity $t \geqslant 4$.

✥ 6. Let $K_{\mathbb{N}}^{(t)}$ denote the (countably) infinite complete t-uniform hypergraph, whose vertex set is $\mathbb{N}$ and whose hyperedges are all t-tuples of distinct integers. Prove the infinite hypergraph Ramsey theorem: for all $t, q \geqslant 2$, and for every q-coloring of $E(K_{\mathbb{N}}^{(t)})$, there is an infinite monochromatic clique.

✥ 7. Prove that for all $q, t \geqslant 2$, there exists an integer N such that the following holds. Identify the vertex set of $K_N^{(t)}$ with $[\![N]\!]$. For any q-coloring of $E(K_N^{(t)})$, there is a set $S \subseteq [\![N]\!]$ such that S forms a monochromatic clique, and $|S| > t + \min S$, where $\min S$ denotes the smallest element of S.

Hint: Although this is a purely "finitary" statement, you should apply the infinite hypergraph Ramsey theorem. In fact, the famous *Paris–Harrington theorem* states that this statement cannot be proved in Peano arithmetic.