1. For every $N \geqslant 3$, give N points in $\mathbb{R}^2$ with no three in convex position. This shows that the assumption in Theorem 10.3.4 that no three points are collinear is necessary.

2. Let $N = r_3(k)$, and let $p_1, \ldots, p_N$ be points in $\mathbb{R}^2$ with no three collinear. Define $\chi : E(K_N^{(3)}) \to \{\text{even}, \text{odd}\}$ by

$$\chi(\{i,j,\ell\}) \coloneqq \begin{cases} \text{even} & \text{if there are an even number of points } p_m \text{ in the triangle } p_i p_j p_\ell, \\ \text{odd} & \text{otherwise.} \end{cases}$$

Prove that a monochromatic $K_k^{(3)}$ under χ corresponds to k points in convex position. Conclude that $\text{Kl}(k) \leqslant r_3(k)$, and in particular obtain a new proof of Theorem 10.3.4.

3. A collection of points in $\mathbb{R}^d$ is said to be *in general position* if no $d+1$ of them lie on a $(d-1)$-dimensional hyperplane. (So in two dimensions, this says that no three are collinear, in three dimensions it says that no four are coplanar, etc.)

   (a) Prove that among any $d+3$ points in $\mathbb{R}^d$ which are in general position, there are $d+2$ in convex position.

   (b) Given $k \geqslant d+2$, let $N = r_{d+2}(d+3, k)$. Prove that among any N points in $\mathbb{R}^d$ in general position, there are k in convex position.

   (c) Prove that among any $\text{Kl}(k)$ points in $\mathbb{R}^d$, no three collinear, there are k in convex position.

   [This is stronger than the result in (b) in two ways: the bound is *independent* of d, and the assumption is weakened from general position to no three collinear.]

4. (a) Let $k, \ell \geqslant 2$. Prove that in any sequence of $(k-1)(\ell-1)+1$ distinct real numbers, there is an increasing subsequence of length k or a decreasing subsequence of length ℓ.

   (b) Prove that the result in (a) is best possible, by finding a sequence of $(k-1)(\ell-1)$ distinct real numbers with no increasing subsequence of length k and no decreasing subsequence of length ℓ.

5. Prove that any sequence of (not necessarily distinct) real numbers of length $(k-1)^3+1$ contains a subsequence of length k that is strictly increasing, strictly decreasing, or constant. Prove that this bound is best possible.

6. Let us say that a coloring of $E(K_k)$ is *semi-starry* if the vertices can be sorted as $v_1, \ldots, v_k$ such that all edges $v_i v_j$, where $j > i$, are of the same color. (The only difference from a starry coloring is that we do not require these colors to be distinct.)

   (a) Prove that if $N \geqslant (k-1)^2+1$, then any semi-starry coloring of $E(K_N)$ contains a monochromatic or starry K_k. Such a result was implicitly used in the proof of Theorem 11.2.2.

(b) Prove that if $N \geqslant k^{4k}$, then any coloring of $E(K_N)$, with an arbitrary number of colors, contains a rainbow or a semi-starry K_k.

(c) Show that there exists a coloring of $E(K_N)$, where $N = k^k$, with no rainbow or semi-starry K_k. Thus, the result of part (b) is best possible up to the constant factor in the exponent.

7. Prove the bipartite canonical Ramsey theorem, which states the following. For every $k \geqslant 2$, there exists some N such that in any coloring of $E(K_{N,N})$, with an arbitrary number of colors, there is a $K_{k,k}$ which is monochromatic, rainbow, or starry.

(Here, a $K_{k,k}$ is *rainbow* if all k^2 edges receive different colors, and is *starry* if it is colored by exactly k distinct colors, each of whose color classes is a star $K_{1,k}$.)

⌖8. (a) Prove that any infinite sequence of distinct real numbers contains an infinite subsequence that is (non-strictly) increasing or (non-strictly) decreasing.

(b) Prove the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence.

⋆9. Prove that $\mathrm{Kl}(5) = 9$.

⋆10. Construct, for every $k \geqslant 4$, a collection of 2^{k-2} points in $\mathbb{R}^2$, no three collinear, with no k of them in convex position.

Hint: Here is a solution for $k = 5$; try to generalize it.