1. (a) Using the fact that $r(k) < 4^k$, prove that $r(k;q) < 4^{4^{4^{\cdot^{\cdot^{\cdot^{k}}}}}}$, where the number of 4s is $\lceil \log_2 q \rceil$.
   (b) Prove Theorem 2.1.5 in the notes. In particular, derive the bound $r(k;q) < q^{qk}$, which is much stronger than that in part (a).

2. Prove that $r(3;q) \leqslant \lceil e \cdot q! \rceil$, where e is Euler's constant.

3. (a) Prove that, for any fixed $k \geqslant 3$, the limit

$$\lim_{q \to \infty} r(k;q)^{1/q}$$

   exists. Conclude that Open problem 2.3.2 from the notes is a well-posed question.

   Hint: Use Fekete's lemma. If you've never heard of Fekete's lemma, look it up and try to prove it before using it!

   ? (b) Prove that, for any fixed $q \geqslant 2$, the limit

$$\lim_{k \to \infty} r(k;q)^{1/k}$$

   exists.

4. The proof of Lemma 3.1.1 in the lecture notes is not 100% correct, as mentioned in Footnote 1. In this problem you will correct this.

   Let G satisfy the assumptions of Lemma 3.1.1, and let $t = N/M$. Let H be a random induced subgraph of G obtained by picking exactly t vertices of G, uniformly at random (i.e. each of the $\binom{N}{t}$ choices is equally likely). Prove that with positive probability, H has no independent set of order k, and hence

$$r(s,k) > t = \frac{N}{M}.$$

5. In the approach using Lemma 3.2.1, we lower-bound $r(k;q)$ by picking $q-2$ random homomorphisms to some K_k-free graph G, and using the last two colors to randomly color all remaining edges. Instead, we could have used $q-r$ random homomorphisms (for some $r < q$), and r random colors for the remaining edges. Prove that picking $r=2$ gives the strongest bounds, hence this extra generality ends up not being useful.

6. Let $f, g_1, \ldots, g_q : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that there exist $\varepsilon, \delta > 0$ such that whenever $x, y \in \mathbb{R}$ satisfy $f(x) - f(y) \geqslant \varepsilon$, then

$$\max_{i \in [\![q]\!]} (g_i(x) - g_i(y)) \geqslant \delta.$$

   Prove that if $g_1, \ldots, g_q$ are all bounded, then f is bounded as well.

$\star$ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

$\star$ 7. Prove that $r(3,3,3) = 17$.

$\star$ 8. Prove Lemma 3.3.2 in the lecture notes. Use it to deduce Theorem 2.3.1, which remains the best known lower bound on $r(k;q)$ for fixed $q \geqslant 3$.

$\maltese$ 9. In this problem, you will see the original approach of Conlon–Ferber to the improved lower bounds on multicolor Ramsey numbers.

For a positive integer t, let $V_t \subseteq \mathbb{F}_2^t$ denote the subspace consisting of all vectors in $\mathbb{F}_2^t$ with an even number of entries equal to 1. Define a graph G_t with vertex set V_t by setting $x \sim y$ if $x \cdot y = 1$, where $x \cdot y = \sum_{i=1}^t x_i y_i$ denotes the usual dot product on $\mathbb{F}_2^t$.

(a) Prove that if t is even, then G_t is K_t-free.

(b) Prove that if t is odd, then G_t is K_{t+1}-free.

(c) Prove that every independent set in G_t is contained in a vector subspace of dimension at most $t/2$.

$\star$ (d) Prove that the number of independent sets in G_t of order at most t is at most $2^{\frac{5}{8}t^2+o(t^2)}$.

(e) Using the facts above and Lemma 3.2.1 from the notes, obtain a new proof that $r(k;q) \geqslant (2^{\frac{3}{8}q-\frac{1}{4}})^{k-o(k)}$ for $q \geqslant 3$.

$\star$ (f) Working with $t = 2k$, and randomly sampling a subset of V_t, obtain a different proof that $r(k;2) \geqslant 2^{\frac{k}{2}-o(k)}$.

? (g) In the proof of (f), you showed that a random induced subgraph of G_t, where $t = 2k$, has no clique or independent set of order k. Can you find an *explicit* description of such a subset (thus resolving Open problem 2.2.3)?