1. Prove that if G is an n-vertex graph with average degree d, then $\alpha(G) \geqslant n/(1+d)$.

2. In the proof of Lemma 4.1.3, we used a number of properties about the function $f(d)$. In this problem you will verify that these properties hold.

   (a) Prove that f istwice differentiable on $(0, \infty)$.

   (b) Prove that $f'(d) \leqslant 0$ for all $d \in (0, \infty)$.

   (c) Prove that $f''(d) \geqslant 0$ for all $d \in (0, \infty)$.

   (d) Prove that f satisfies the differential equation

$$(d+1)f(d) = 1 + (d - d^2)f'(d).$$

3. Let n be an integer and let $0 \leqslant d \leqslant n$ be a real number. Consider a random n-vertex graph G formed by including each edge independently with probability d/n.

   (a) Prove that if $d = \omega(1)$, then with probability $1 - o(1)$, we have

$$\alpha(G) \leqslant (1 + o(1))\frac{2n \ln d}{d}.$$

   (b) Prove that if $d = o(n^{1/3})$, then G is triangle-free with probability $1 - o(1)$.

   (c) Prove that if $d = \omega(1)$, the average degree of G is $(1 + o(1))d$ with probability $1 - o(1)$.

   Conclude that Lemma 4.1.3 is best possible up to a factor of $2 + o(1)$.

⋆4. Prove that, for any fixed $s \geqslant 3$, we have

$$r(s,k) = O_s\left(\frac{k^{s-1}}{(\log k)^{s-2}}\right).$$

5. (a) Prove that, for any fixed $s \geqslant 3$, we have

$$r(s,k) \geqslant k^{\frac{s-1}{2} - o(1)},$$

   where the $o(1)$ term tends to 0 as $k \to \infty$.

   ⋆ (b) Improve the exponent to $\frac{s}{2} - o(1)$.

   ⋆⋆ (c) Improve the exponent to $\frac{s+1}{2} - o(1)$.

   ? (d) Improve the exponent to $\frac{s+1}{2} + \varepsilon - o(1)$, for any $s \geqslant 5$ and any $\varepsilon > 0$.

⋆ means that a problem is hard.

? means that a problem is open.

✥ means that a problem is on a topic beyond the scope of the course.

$\oplus\star$6. Prove that an n-vertex C_4-free graph with average degree d has independence number at least

$$(1 - o(1))\frac{n \ln d}{d},$$

where the $o(1)$ term tends to 0 as $d \to \infty$.

Hint: Consider the function

$$g(x) := \int_0^1 \frac{\sqrt{1-t}}{2 + (x-2)t} \, \mathrm{d}t.$$

$\oplus$7. In this problem, you will give an alternative proof of Lemma 4.1.3 (albeit with a worse constant factor). Let G be an n-vertex triangle-free graph with average degree d, and assume that $d \geqslant 16$. Let S be a uniformly random independent set in G.

(a) For every vertex $v \in V(G)$, let X_v be the indicator random variable for the event $v \in S$. Let Y_v be the random variable counting how many neighbors of v are in S. Prove that

$$\sum_{v \in V(G)} (dX_v + Y_v) \leqslant 2d|S|.$$

(Note that both sides of this inequality are random quantities—the statement is that this inequality is valid regardless of the random outcome.)

$\star$(b) Prove that

$$\mathbb{E}\left[\sum_{v \in V(G)} (dX_v + Y_v)\right] \geqslant \frac{\log d}{4}.$$

(c) Prove that G has an independent set of order at least $n \log d/(8d)$.

$\oplus$8. Recall that $\theta(d)$ denotes the maximum sphere packing density in $\mathbb{R}^d$. Prove that $\theta(d) \geqslant 2^{-d}$.