1. Prove that, for any $r \geqslant 1$,

$$r(\underbrace{3, 3, \ldots, 3}_{r \text{ times}}, k) = \Omega\left(\frac{k^{r+1}}{(\log k)^{C_r}}\right),$$

for some constant C_r depending only on r. Note that this matches the upper bound of Theorem 2.1.5 up to the logarithmic factor.

⋆2. Prove that the graph Λ_q has no O'Nan configuration.

3. Let q be a prime power. Construct a graph Π_q with vertex set $V(\Pi_q) = \mathbb{F}_q^2$, in which two vertices $(x_1, y_1), (x_2, y_2)$ are adjacent if and only if $x_1x_2 + y_1y_2 = 1$.

   (a) Prove that Π_q is C_4-free.

   ⋆(b) Prove that Π_q satisfies the assumptions of Lemma 4.3.1 with $\beta = \Theta(1/q)$ and $R = \Theta(q^{3/2})$.

   Remark: You should feel free to prove this with logarithmic losses in the value of R. The only way I know how to prove this involves techniques (from spectral graph theory) that we will not cover in this class, but I believe there should be an "elementary" proof. Please let me know if you find one!

4. Let q be a prime power. We define a graph $\Gamma_q^{(5)}$ to be the natural five-dimensional analogue of Γ_q. Namely, $\Gamma_q^{(5)}$ is a bipartite graph with parts $P \cup L$, where P is identified with $\mathbb{F}_q^5$, and L comprises all lines in $\mathbb{F}_q^5$ whose direction is of the form $(1, z, z^2, z^3, z^4)$ for some $z \in \mathbb{F}_q$.

   (a) Prove that $\Gamma_q^{(5)}$ is C_4-free and C_{10}-free.

   (b) Define a natural analogue $G_q^{(5)}$ of G_q. It is natural to suppose that $G_q^{(5)}$ is C_5-free with probability 1; show that this is *not* the case.

   Hint: Show that $\Gamma_q^{(5)}$ is *not* C_8-free. Use this to find a C_5 in $G_q^{(5)}$.

5. Verify that Lemmas 4.3.8 and 4.3.9 imply Theorem 4.3.10.

✥6. A *subdivision* of a graph H is obtained from H by replacing every edge of H by a path of some length (not necessarily the same length for all edges, and paths of length 1 are allowed, so that H is a subdivision of itself). A famous conjecture of Hajós asserts that if $\chi(G) \geqslant k$, then G contains a subdivision of K_k as a subgraph.

   (a) Prove that Hajós' conjecture is true for $k \leqslant 3$.

   ⋆(b) Prove that Hajós' conjecture is true for $k = 4$.

⋆ means that a problem is hard.
? means that a problem is open.
✥ means that a problem is on a topic beyond the scope of the course.

(c) Prove that Hajós' conjecture for $k = 5$ implies the four-color theorem. Conclude that it is probably pretty hard to prove the $k = 5$ case.

(d) Prove that if Hajós' conjecture is true, then $r(k) \leqslant 3k^3$. Conclude that Hajós' conjecture is false.

(e) Prove that if Hajós' conjecture is true, then $r(3, k) \leqslant 12k$. Conclude that Hajós' conjecture is false.

✧ 7. A classical fact in graph theory is that there exist triangle-free graphs of arbitrarily high chromatic number. A standard proof, taught in most introductory graph theory courses, uses the *Mycielski construction*. In this exercise, you will see two alternative Ramsey-theoretic proofs.

(a) For an integer N, let S_N be a graph with vertex set $\binom{[\![N]\!]}{2}$, where we think of the vertices of S_N as ordered pairs (a, b) with $1 \leqslant a < b \leqslant N$. The edges of S_N consist of all pairs of the form $((a, b), (b, c))$ for $a < b < c$. Prove that S_N is triangle-free, and that $\chi(S_N) \to \infty$ as $N \to \infty$.

(b) The graph G_q constructed in class is triangle-free; prove that $\chi(G_q) \to \infty$ as $q \to \infty$.

✧ 8. (a) Let $K_{\mathbb{N}}$ denote the complete graph whose vertex set is $\mathbb{N}$. Prove the "infinite Ramsey theorem": for any positive integer q, and any q-coloring of $K_{\mathbb{N}}$, there is an infinite monochromatic clique.

⋆ (b) Prove that the finite and infinite Ramsey theorems are equivalent.

Hint: This fact is often called "compactness", and you may want to use something else called compactness in the proof.

✧ 9. Prove that there is an infinite set $S \subseteq \mathbb{N}$ such that for every $a, b \in S$, the number $a + b$ has an even number of prime factors (counted without multiplicity).