1. (a) Prove that $r(T;q) = O(qn)$ for every $q \geqslant 2$ and every n-vertex tree T.
   $\star$(b) Prove that $r(T;q) = \Theta(qn)$ for every $q \geqslant 2$ and every n-vertex tree T.

2. Prove that the off-diagonal graph Ramsey number $r(C_4, K_k)$ satisfies the bounds
$$c\frac{k^{\frac{3}{2}}}{\log k} \leqslant r(C_4, K_k) \leqslant C\frac{k^2}{(\log k)^2}$$
for some absolute constants $c, C > 0$.

   Hint: You may assume the results of exercise 6 from Homework #3 and exercise 3 from Homework #4.

3. Prove that $r(K_{1,k}) = 2k$ if k is odd, and $r(K_{1,k}) = 2k - 1$ if k is even.

4. Let kK_2 denote a matching with k edges, that is, a disjoint union of k copies of the single-edge graph K_2. Prove that $r(kK_2) = 3k - 1$ for all $k \geqslant 1$.

$\star$5. Let P_k denote a k-vertex path. Prove that for all $k \geqslant \ell \geqslant 2$,
$$r(P_k, P_\ell) = k + \left\lfloor \frac{\ell}{2} \right\rfloor - 1.$$

6. Prove that for every integer k and for every n-vertex tree T, we have
$$r(K_k, T) = (k-1)(n-1) + 1.$$

7. Prove that the two definitions of degeneracy in Definition 5.4.3 are equivalent.

8. Prove that there exist absolute constants $C, c > 0$ such that the following holds for all n. There exists an n-vertex graph H with degeneracy $d \geqslant c \log n$ and $r(H) \leqslant Cn$.

   Note that this result is close to optimal; by Theorem 5.4.4, such an upper bound on $r(H)$ cannot hold if $c > 2$.

9. (a) Prove that every 2-coloring of $[\![N]\!]^2$ contains a monochromatic $k \times k$ subgrid, where $k = \Omega(\log N)$. Here, a *subgrid* is a collection of points $\{(x_i, y_j)\}_{i,j=1}^k$, for some $x_1, \ldots, x_k, y_1, \ldots, y_k \in [\![N]\!]$.
   (b) Prove that the 2-coloring in part (a) is somewhat of a red herring. Namely, show that if $S \subseteq [\![N]\!]^2$ has size $|S| \geqslant N^2/2$, then S contains a $k \times k$ subgrid, where $k = \Omega(\log N)$. Thus, part (a) simply follows by letting S consist of the larger color class.

$\star$ means that a problem is hard.
? means that a problem is open.
$\oplus$ means that a problem is on a topic beyond the scope of the course.

(c) Prove that the result in part (a) is tight up to a constant factor, in the sense that there exists an absolute constant $C > 0$, as well as a 2-coloring of $[\![N]\!]^2$ with no monochromatic $k \times k$ subgrid, where $k = C \log n$.

$\star$ 10. Prove that

$$2^q k < r(C_{2k+1}; q) \leqslant C(q+2)!k,$$

for some absolute constant C.

$\star\star$ 11. Prove that $r(K_{k,k}) = O(2^k \log k)$.

✥ 12. In a red/blue coloring of $E(K_N)$, denote by $\deg_R(v), \deg_B(v)$ the red and blue degrees, respectively, of a vertex v.

(a) Prove that the number of monochromatic triangles in such a coloring is equal to

$$\sum_{v \in V(K_N)} \left[\binom{\deg_R(v)}{2} + \binom{\deg_B(v)}{2} \right] - \binom{N}{3}. \qquad (*)$$

(b) Prove that every 2-coloring of $E(K_6)$ contains at least two monochromatic triangles, and in particular obtain a new proof that $r(3) \leqslant 6$.

(c) As a function of N, what is the minimum number of monochromatic triangles in a 2-coloring of $E(K_N)$?

$\star$ (d) Prove that no analogue of $(*)$ is true for cliques of order $k \geqslant 4$, or if the number of colors is $q \geqslant 3$. That is, in either of these settings, the number of monochromatic copies of K_k cannot be expressed only in terms of the vertex degrees.

✥ 13. For a set $S \subseteq \mathbb{Z}^2$, let us define its *k-square density* to be

$$d_k(S) := \max_{\substack{A,B \subseteq \mathbb{Z} \\ |A|=|B|=k}} \frac{|S \cap (A \times B)|}{|A||B|}.$$

and its *silly density* to be

$$d_{\text{silly}}(S) := \lim_{k \to \infty} d_k(S).$$

Prove that the limit defining $d_{\text{silly}}(S)$ exists, and that for every $S \subseteq \mathbb{Z}^2$, we have that $d_{\text{silly}}(S) = 0$ or $d_{\text{silly}}(S) = 1$. Conclude that this is a pretty silly way to define the density of a set.