1. Prove the general dependent random choice lemma, Lemma 5.4.11.

2. Recall that Q_d denotes the d-dimensional hypercube graph, with $n = 2^d$ vertices.

   (a) Prove that Q_d is a bipartite graph with maximum degree d on one side. Note that we cannot directly apply Theorem 5.4.10, since n depends on d and Theorem 5.4.10 requires n to be sufficiently large. However, if Theorem 5.4.10 is valid, check that it implies $r(Q_d) \leqslant n2^{5d\sqrt{\log n}} = 2^{d+5d^{3/2}} = n^{\Theta(\sqrt{\log n})}$.

   (b) By applying Lemma 5.4.11 and being more careful, prove that $r(Q_d) \leqslant 2^{3d} = n^3$. Note that this bound is polynomial in n, whereas the bound in part (a) is superpolynomial in n.

3. Let $\widehat{K_k}$ denote the 1-*subdivision* of K_k. This is a graph on $k + \binom{k}{2}$ vertices, obtained by introducing a new vertex in the middle of every edge of K_k. Equivalently, it is obtained from K_k by replacing every edge by a 2-edge path.

   (a) Prove that $\widehat{K_k}$ is a bipartite graph with maximum degree 2 on one side. Conclude from Theorem 5.4.10 that $r(\widehat{K_k}) \leqslant k^2 2^{15\sqrt{\log k}}$.

   (b) By applying Lemma 5.4.11 and being more careful, prove that $r(\widehat{K_k}) = O(k^2)$. Note that this bound is tight up to the implicit constant since $\widehat{K_k}$ has $\Theta(k^2)$ vertices.

⋆4. Prove that for every $d \geqslant 1$, there exists $C > 0$ such that the following holds for sufficiently large n. If H is a d-degenerate bipartite graph, then $r(H) \leqslant n2^{C(\log n)^{2/3}} = n^{1+o(1)}$.

5. Let G be an ε-quasirandom graph. Prove that for all disjoint $S, T \subseteq V(G)$ with $|S|, |T| \geqslant \varepsilon |V(G)|$, we have $|d(S,T) - d(G)| \leqslant 2\varepsilon$.

6. Prove the embedding lemma, Lemma 6.1.3, in the case $H = K_3$.

   Don't worry too much about the exact assumptions $d(G) \geqslant (2\Delta\varepsilon)^{1/\Delta}$ and $N \geqslant 2n/\varepsilon$—it's OK if you prove this under stronger assumptions of a similar flavor.

7. Prove Theorem 6.2.3, the linear bound on multicolor Ramsey numbers of bounded-degree graphs.

✣⋆8. Prove that one cannot do better than exponential bounds in the ε-quasirandom set lemma, Lemma 6.1.4. More precisely, show that there is an absolute constant $c > 0$ such that for every $\varepsilon > 0$ and every sufficiently large N, there exists an N-vertex graph whose largest ε-quasirandom induced subgraph has at most $2^{-\varepsilon^{-c}}N$ vertices.

✣9. The proof of Lemma 6.1.4 is not optimized quantitatively.

   (a) What is the strongest bound on δ you can obtain by being more careful in the proof?

? (b) Prove that, for some absolute constant $C > 0$, one can take $\delta = 2^{-\varepsilon^{-C}}$ in Lemma 6.1.4. Note that by Exercise 8, this would be best possible up to the value of C.

⊕ 10. In this problem you will construct an *explicit* ε-quasirandom graph.

(a) Fix an odd prime p. Prove that for any $T \subseteq \mathbb{Z}/p\mathbb{Z}$, we have that

$$\sum_{z \in \mathbb{Z}/p\mathbb{Z}} \left| \sum_{t \in T} e^{2\pi i t z/p} \right|^2 = p|T|.$$

(b) Let $\chi : \mathbb{Z}/p\mathbb{Z} \to \{-1, 0, 1\}$ be the *quadratic character* mod p, namely the function

$$\chi(x) = \begin{cases} 1 & \text{if } x \text{ is a quadratic residue mod } p, \\ -1 & \text{if } x \text{ is a quadratic non-residue mod } p, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that if $p \equiv 1 \pmod 4$, then $\chi(x) = \chi(-x)$ for all $x \in \mathbb{Z}/p\mathbb{Z}$.

(c) Prove the *Gauss sum formula*,

$$\left| \sum_{z \in \mathbb{Z}/p\mathbb{Z}} \chi(z) e^{2\pi i z/p} \right| = \sqrt{p}.$$

(d) Prove that for all $X, Y \subseteq \mathbb{Z}/p\mathbb{Z}$, we have that

$$\left| \sum_{x \in X} \sum_{y \in Y} \chi(x - y) \right| \leqslant \sqrt{p|X||Y|}.$$

(e) Let $p \equiv 1 \pmod 4$ be prime. Define the following graph, called the *Paley graph* P_p, whose vertex set is $\mathbb{Z}/p\mathbb{Z}$. For vertices a, b, we join them by an edge if and only if $b - a$ is a quadratic residue mod p; note that by part (b) this is indeed a well-defined graph.

Fix some $\varepsilon > 0$. Prove that if p is sufficiently large with respect to ε, then P_p is ε-quasirandom.