1. (a) Fix a q-coloring $\chi_0 : E(K_n) \to [\![q]\!]$. Prove that for every $\sigma > 0$, there exists $\delta > 0$ such that the following holds. If a coloring $\chi : E(K_N) \to [\![q]\!]$ does not contain χ_0 as an induced subcoloring, then there exists a set $S \subseteq V(K_N)$ with $|S| \geqslant \delta N$ and an index $i \in [\![q]\!]$, such that at most $\sigma\binom{|S|}{2}$ of the edges in S are colored by color i under χ.

   (b) Prove that the $q = 2$ case of part (a) is equivalent to Rödl's theorem, Theorem 6.3.3.

   (c) You might have expected the multicolor generalization of Rödl's theorem to say that *all colors but one* have edge density at most σ in S. Prove that such a statement is false, even in the case $n = q = 3$.

   More precisely, show that there is a $E(K_N) \to [\![3]\!]$ such that no triangle receives all three colors, but such that every linearly-sized subset has edge density at least $\frac{1}{3}$ in at least two of the colors.

✣2. Let G be a random N-vertex graph, where each edge is included independently with probability $\frac{1}{2}$. Prove that, with positive probability, G has the following property for sufficiently large N, where $C > 0$ is an absolute constant. Every subset $S \subseteq V(G)$ with $|S| \geqslant C \log N$ satisfies $\frac{1}{3} \leqslant d(S) \leqslant \frac{2}{3}$.

   This result shows that if we drop the assumption that G is induced-H-free, then nothing like Theorem 6.3.3 could possibly be true.

3. A graph H is said to have the *Erdős–Hajnal property* if there exists $\varepsilon > 0$, depending only on H, such that every induced-H-free N-vertex graph has a clique or an independent set of size at least N^ε. Recall that the Erdős–Hajnal conjecture asserts that all graphs have the Erdős–Hajnal property.

   (a) Prove that if $H = K_k$ is a complete graph, then H has the Erdős–Hajnal property.

   (b) Prove that if H has the Erdős–Hajnal property, then so does its complement graph $\overline{H}$.

   ⋆(c) Let P_4 denote the four-vertex path graph. Prove that if G is an induced-P_4-free graph, then either G or $\overline{G}$ is disconnected. Using this, prove that P_4 has the Erdős–Hajnal property.

4. A graph G is *minimally Ramsey* for H if G is Ramsey for H, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H. H is called *Ramsey finite* if there are only a finite number of minimally Ramsey graphs for H, and *Ramsey infinite* otherwise.

   (a) Let $G = K_3 * C_\ell$, where $\ell \geqslant 3$ is odd. Prove that G is minimally Ramsey for K_3. Conclude that K_3 is Ramsey infinite.

   (b) Determine the set of Ramsey minimal graphs for $K_{1,2}$.

   ⋆(c) Prove that $K_{1,k}$ is Ramsey finite if and only if k is odd.

   ⋆(d) Prove that every tree which is not a star is Ramsey infinite.

5. A graph G is called *q-minimally Ramsey* for a graph H if G is Ramsey for H in q colors, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H in q colors.
   (a) Prove that if G is q-minimally Ramsey for H, then every edge of G lies in at least q copies of H.
   (b) Prove that if G is q-minimally Ramsey for H, then G has at least $q^{e(H)-1}$ copies of H.
   (c) Prove Proposition 7.1.9.

6. Prove that for every $n, q \geqslant 2$, there exists some N such that $K_{N,N}$ is q-color induced Ramsey for $K_{n,n}$.

✥7. (a) Prove that for any $\ell \geqslant 4$, the cycle C_ℓ is a subgraph of a triangle tree (and hence Ramsey obligatory for K_3).
   (b) Prove that K_4 is not a subgraph of any triangle tree.

✥⋆8. Prove Theorem 7.1.3 from Theorem 7.1.5. You should in fact assume the following strengthening of Theorem 7.1.5; in the same setup as in the theorem statement, we have that

$$\Pr(G \text{ is Ramsey for } H \text{ in } q \text{ colors}) \begin{cases} \geqslant 1 - e^{-cpN^2} & \text{if } p \geqslant CN^{-1/m_2(H)}, \\ \leqslant e^{-cpN^2} & \text{if } p \leqslant cN^{-1/m_2(H)}. \end{cases}$$

Hint: Use Harris's inequality (or the FKG inequality).