1. Determine $\mathrm{HJ}(2; q)$ for every $q \geqslant 1$.

2. (a) Prove that, for every $k, q \geqslant 2$, there exists some N such that any q-coloring of $[\![N]\!]$ contains a k-term *geometric* progression. That is, there exist numbers a, r with $r \geqslant 2$ such that
$$a, ar, ar^2, \dots, ar^{k-1}$$
all receive the same color.

   Hint: This is a one-line corollary of van der Waerden's theorem.

   (b) Prove the following multiplicative analogue of Theorem 9.3.1. For every $m, q \geqslant 2$, there exists N such that in any q-coloring of $[\![N]\!]$, there exist distinct $x_1, \dots, x_m \in [\![N]\!]$ such that all the subset products $\prod_{i \in I} x_i$, for $\varnothing \neq I \subseteq [\![m]\!]$, receive the same color.

   ? (c) Prove the following "combined" version of (b) and Theorem 9.3.1. For every $q \geqslant 2$, there exists N such that for any q-coloring of $[\![N]\!]$, there exist x, y such that
$$x, y, x + y, xy$$
all receive the same color.

3. (a) Prove that for every $k \geqslant 3$, there exists some N such that the following holds. In any 2-coloring of $[\![N]\!]$, there exists a k-AP such that all its terms, *as well as its common difference*, receive the same color. That is, there exist $a, r \in [\![N]\!]$ such that
$$r, a, a + r, a + 2r, \dots, a + (k - 1)r$$
all receive the same color.

   Hint: Begin by applying van der Waerden's theorem to find a monochromatic $(k^2 + 1)$-AP.

   (b) Prove a multicolor generalization of part (a). That is, for any $k, q \geqslant 3$, there exists some N such that in any q-coloring of $[\![N]\!]$, there exists a k-AP such that all its terms, as well as its common difference, receive the same color.

4. (a) By coloring randomly, prove that
$$W(k; q) > q^{\frac{k-1}{2}}.$$

   (b) Prove that there are at most $N^2/(2(k - 1))$ arithmetic progressions of length k in $[\![N]\!]$. Using this, improve your bound in (a) to
$$W(k; q) > \sqrt{2(k - 1)}q^{\frac{k-1}{2}}.$$

$\star$ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

⌖ 5. In d-dimensional tic-tac-toe, two players take turns putting an X or an O in one of the positions of the d-dimensional grid $[\![3]\!]^d$. A player wins when she constructs a line consisting entirely of her symbol. Prove that if d is sufficiently large, then the first player has a winning strategy.

⌖ 6. In this problem, you will see some better lower bounds on van der Waerden numbers, using some properties of finite fields. Let p be prime, and consider the finite field $\mathbb{F}_{2^p}$. View $\mathbb{F}_{2^p}$ as a vector space over $\mathbb{F}_2$, and let A be any codimension-one subspace of this vector space.

(a) Prove that A does not contain p elements in geometric progression, that is, there do not exist $a, r \in \mathbb{F}_{2^p}$ with $a \neq 0$ and $r \notin \{0, 1\}$ such that

$$a, ar, \ldots, ar^{p-1} \in A.$$

(b) Let $B = \mathbb{F}_{2^p} \setminus A$. Prove that B does not contain $p+1$ elements in geometric progression, that is, there do not exist $a, r \in \mathbb{F}_{2^p}$ with $a \neq 0$ and $r \notin \{0, 1\}$ such that

$$a, ar, \ldots, ar^{p-1}, ar^p \in B.$$

(c) Using the fact that the multiplicative group of $\mathbb{F}_{2^p}$ is cyclic, conclude from the above that

$$W(p+1; 2) > 2^p - 1.$$

Note that this is substantially better than the bounds in problem 4 in the case that $q = 2$ and that $k - 1$ is prime.

⋆ (d) Using more cleverly that the multiplicative group of $\mathbb{F}_{2^p}$ is cyclic, prove that

$$W(p+1; 2) > p(2^p - 1).$$

? (e) Extend the above to work for all k, not just those that are one more than a prime. Namely, prove that

$$W(k; 2) = \Omega(k2^k)$$

for all k.

⌖ 7. (a) Prove that van der Waerden's theorem is equivalent to the following statement: in any coloring of $\mathbb{N}$ with a finite number of colors, there are monochromatic arithmetic progressions of every finite length.

(b) Construct a 2-coloring of $\mathbb{N}$ with no *infinite* monochromatic arithmetic progression, thus showing that the statement in (a) is best possible.

⋆⋆ (c) Prove that in any finite coloring of $\mathbb{N}$, there is an infinite set A such that all finite sums of distinct elements of A receive the same color.