1. (a) Prove that every 2-coloring of $E(K_N)$ contains a monochromatic N-vertex tree.
   (b) Prove that for every $q \geqslant 2$, there exists some $\delta > 0$ such that the following holds. In any q-coloring of $E(K_N)$, one of the color classes contains *all* of the trees on δN vertices.

Solution.

(a) We prove this by induction on N, with the base case $N = 1$ being trivial since no edges are colored and the only 1-vertex tree has no edges. Now suppose the result has been proved for $N - 1$, and consider a 2-coloring of $E(K_N)$. Fix a vertex $v \in V(K_N)$. If all edges incident to v have the same color, then we have found a monochromatic copy of the N-vertex tree $K_{1,N-1}$, and we are done.

So we may assume that v is incident to at least one red and at least one blue edge, say vu is red and vw is blue. Now consider the induced coloring on $V(K_N) \setminus \{v\}$. By the induction hypothesis, there is a monochromatic $(N-1)$-vertex tree T in this coloring. If it is red, then we may add v as a leaf to T, attached to u, and obtain a red N-vertex tree. Similarly, if T is blue, then we may add v as a leaf attached to w to obtain a blue N-vertex tree.

(b) There is nothing to prove if $N = 1$, so we may assume $N \geqslant 2$. We claim that this result is true with $\delta = \frac{1}{4q}$. Indeed, fix a q-coloring of $E(K_N)$. At least one of the color classes, say red, has at least $\frac{1}{q}\binom{N}{2}$ edges. If we let G be the graph of red edges, this implies that the average degree in G is at least $\frac{1}{q}(N-1) \geqslant \frac{N}{2q}$, where we use our assumption $N \geqslant 2$ to conclude that $N - 1 \geqslant \frac{N}{2}$. By Lemma 5.2.2, this implies that G has a subgraph G' with minimum degree at least $\frac{N}{4q}$.

We now claim that every tree on δN vertices is a subgraph of G', and thus of G, which means that the red color contains all all of the trees on δN vertices. To prove this, fix a tree T on $\delta N = \frac{N}{4q}$ vertices. By Lemma 5.2.3, T is a subgraph of G', as claimed. □

2. Prove that the q-color Ramsey number $r(2, 3, \ldots, q, q+1)$ satisfies the bounds

$$2^{cq^2} \leqslant r(2, 3, \ldots, q, q+1) \leqslant q^{Cq^2}$$

for some absolute constants $c, C > 0$.

Solution. Let $R = r(2, 3, \ldots, q, q+1)$. By the monotonicity of Ramsey numbers, we have that $R \leqslant r(q+1; q)$. Therefore, by Theorem 2.1.5, we have

$$R \leqslant r(q+1; q) \leqslant q^{q(q+1)} \leqslant q^{2q^2},$$

which proves the upper bound with $C = 2$.

For the lower bound, we also apply monotonicity, as follows. Note that at least $\lfloor q/2 \rfloor$ of the numbers $2, 3, \ldots, q+1$ are at least $\lfloor q/2 \rfloor$. Therefore, $R \geqslant r(\lfloor q/2 \rfloor; \lfloor q/2 \rfloor)$. By Proposition 2.2.5, we conclude that

$$R \geqslant 2^{\lfloor q/2 \rfloor \lfloor \lfloor q/2 \rfloor /2 \rfloor}.$$

Suppose for the moment that $q \geqslant 4$, which implies that $\lfloor q/2 \rfloor \geqslant q/3$ and that $\lfloor \lfloor q/2 \rfloor /2 \rfloor \geqslant q/7$. Then we conclude that for $q \geqslant 4$, we have

$$R \geqslant 2^{\lfloor q/2 \rfloor \lfloor \lfloor q/2 \rfloor /2 \rfloor} \geqslant 2^{(q/3)(q/7)} = 2^{cq^2},$$

where $c = \frac{1}{21}$. On the other hand, if $q \leqslant 3$, then we still have the bound $R \geqslant r(2,3) = 3 \geqslant 2^{q^2/9}$. So in either case we conclude that $R \geqslant 2^{cq^2}$, where $c = \frac{1}{21}$. □

3. Prove that if N is sufficiently large, then the following holds. Among any N points in the plane, there are three of them that determine an angle greater than 179°.

Solution. Recall that in any k-gon, the sum of the internal angles equals $(k-2)\cdot 180°$. Moreover, if this k-gon is convex, then every internal angle in it is at most 180°. Therefore, in any convex k-gon, one of the internal angles is between $\frac{k-2}{k}\cdot 180°$ and 180°.

Now pick k sufficiently large so that $180\frac{k-2}{k} > 179$, and let $N = \mathrm{Kl}(k)$. Fix N points in the plane. If any three of them are collinear, then they give an angle of $180° > 179°$, so we are done, and we may assume that no three are collinear. By Theorem 10.3.4, there are k of these points in convex position, meaning that they form the vertices of a convex k-gon. By the above, this means that one of the internal angles is greater than 179°, completing the proof. □

Alternate solution. Let ℓ be a line in the plane passing through the origin, and denote by $\theta(\ell) \in [0, 180)$ the angle it makes with the x-axis. For a line ℓ not passing through the origin, we define $\theta(\ell) = \theta(\ell')$, where ℓ' is the unique line parallel to ℓ and passing through the origin.

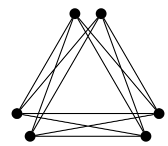
Let ℓ_1 and ℓ_2 be intersecting lines. Then the angles formed at their intersection are exactly $|\theta(\ell_1) - \theta(\ell_2)|$ and $180 - |\theta(\ell_1) - \theta(\ell_2)|$.

Now, let $N = r(3; 360)$, and fix N points $p_1, \ldots, p_N$ in the plane. We define a 360-coloring of $E(K_N)$ as follows. We identify $V(K_N)$ with $\{p_1, \ldots, p_N\}$, and color define $\chi(\{p_i, p_j\}) := \lfloor 2\theta(\ell_{ij})\rfloor$, where ℓ_{ij} is the line between the points p_i, p_j. That is, we color the edge $p_i p_j$ in one of 360 colors, based on whether $\theta(\ell_{ij}) \in [0°, .5°), [0.5°, 1°), \ldots$.

By the choice of N, there is a monochromatic triangle in this coloring, say with vertices p_1, p_2, p_3. This implies that $\theta(\ell_{12}), \theta(\ell_{13}), \theta(\ell_{23})$ are all nearly equal, in that they all lie in the same interval $[x, x+\frac{1}{2})$ for some x. In particular, we have that $|\theta(\ell_{12}) - \theta(\ell_{13})| < 0.5°$, and the same for the other three pairs.

In the triangle formed by p_1, p_2, p_3, all internal angles are the angles of intesection of two of these lines. By the above, we conclude that each internal angle is either less than 1° or greater than 179°. Since these three angles must sum to 180°, we conclude that one of them must be greater than 179°. □

4. Let G be a graph. The *s-blowup* of G, denoted $G[s]$, is the graph obtained by replacing each vertex of G by s vertices, and replacing each edge of G by a complete bipartite graph $K_{s,s}$. For example, here is a picture of $K_3[2]$:



   (a) Prove that for every $s \geqslant 2$, there exists some $N = N(s)$ such that $K_6[N]$ is Ramsey for $K_3[s]$.

   (b) Prove that $N(s) > 2^s$ for all $s \geqslant 4$.

Solution.

(a) We will repeatedly apply the following lemma: for every k, there is some N such that $K_{N,N}$ is Ramsey for $K_{k,k}$. Indeed, in any two-coloring of $E(K_{N,N})$, one of the color classes must have at least $N^2/2$ edges, which is greater than $k^{1/k}(2N)^{2-1/k} + kN$ for N sufficiently large in terms of k. Therefore, by Theorem 5.3.2, this color class must contain a copy of $K_{k,k}$.

Now pick a sequence of numbers $N_1, \ldots, N_{15}$ as follows. We set $N_1 = s$. Having defined N_i, we set N_{i+1} to be sufficiently large so that $K_{N_{i+1},N_{i+1}}$ is Ramsey for K_{N_i,N_i}; such an N_{i+1} exists by the lemma above, hence we can define $N_1, \ldots, N_{15}$. Let $N = N_{15}$; we claim that $K_6[N]$ is Ramsey for $K_3[s]$. To prove this, we fix a 2-coloring of $E(K_6[N])$.

Arbitrarily order the $\binom{6}{2} = 15$ edges of K_6, and do the following. For the first edge in this ordering, we consider the coloring of the corresponding $K_{N,N}$ in $K_6[N]$. By our choice of $N = N_{15}$, we can find here a monochromatic $K_{N_{14},N_{14}}$. We now restrict two of the six parts to only be the vertex set of this $K_{N_{14},N_{14}}$, and arbitrarily shrink the other four parts to subsets of size N_{14}, so we obtain a copy of $K_6[N_{14}]$ in which one of the 15 copies of $K_{N_{14},N_{14}}$ is monochromatic. We now do the same with the next edge in our ordering, and thus restrict to a copy of $K_6[N_{13}]$ where two of the pairs are monochromatic.

Continuing in this fashion, we eventually find within our original coloring a copy of $K_6[s]$ in which all copies of $K_{s,s}$ are monochromatic, but perhaps not with the same color. Now, we apply the fact that K_6 is Ramsey for K_3 to find three parts such that the three $K_{s,s}$ among them *do* have the same color, giving us our monochromatic $K_3[s]$.

(b) Let $N = 2^s$. Consider a uniformly random 2-coloring of $K_6[N]$. Any given copy of $K_3[s]$ has $3s^2$ edges, and hence a probability of 2^{1-3s^2} of being monochromatic. Moreover, the number of copies is at most $\binom{6N}{3s}$, hence by the union bound, the probability that some coloring is monochromatic is at most

$$\binom{6N}{3s}2^{1-3s^2} < \frac{2 \cdot 6^{3s}}{(3s)!}N^{3s}2^{-3s^2} \leqslant (N2^{-s})^{3s} = 1,$$

by our choice of $N = 2^s$, where we use the fact that $2 \cdot 6^{3s} \leqslant (3s)!$ for all $s \geqslant 4$. Hence there exists a coloring of $K_6[N]$ with no monochromatic $K_3[s]$, showing that $N(s) > 2^s$. $\square$

5. Let $k \geqslant 3$ and $N \geqslant 3k$ be integers. Recall that k-AP is short for k-term arithmetic progression.

   (a) Prove that there are at least $N^2/(6k)$ distinct k-APs in $[\![N]\!]$.

   (b) Let $A \subseteq [\![N]\!]$. Prove that A intersects at most $\binom{k}{2}|A|^2$ k-APs in more than one point.

   (c) Prove the *canonical van der Waerden theorem*, which states the following. For every $k \geqslant 3$, there exists some N such that the following holds. In any coloring of $[\![N]\!]$ with an arbitrary number of colors, there is a monochromatic or rainbow k-AP.

Solution.

(a) For every $1 \leqslant a \leqslant \lceil N/2 \rceil$ and every $1 \leqslant r \leqslant \lfloor N/(2k) \rfloor$, the k-AP $a, a+r, \ldots, a+(k-1)r$ is fully contained within $[\![N]\!]$, and all of these k-APs are distinct (since a k-AP is uniquely determined by its starting point and its common difference). There are $\lceil N/2 \rceil \geqslant N/2$ choices for a and $\lfloor N/(2k) \rfloor \geqslant N/(3k)$ choices for r, hence at least $N^2/(6k)$ distinct k-APs in total.

(b) First observe that every pair of integers $a, b \in [\![N]\!]$ lies in at most $\binom{k}{2}$ distinct k-APs. Indeed, once we fix which position a and b have in the k-AP, we have determined the common difference and the starting point of the k-AP.

Now, given a k-AP which intersects A in at least two points, fix two such points $a, b \in A$. There are at most $|A|^2$ choices for a, b, and at most $\binom{k}{2}$ choices for the k-AP containing them, yielding the claimed bound of $\binom{k}{2}|A|^2$.

(c) Set $\delta = 1/(6k\binom{k}{2})$. Let N be sufficiently large so that every $A \subseteq [\![N]\!]$ with $|A| \geqslant \delta N$ contains a k-AP; such an N exists by Theorem 9.7.1. By potentially making N larger, we can also assume that $N \geqslant 3k$. We claim that this choice of N suffices.

To see this, fix an arbitrary coloring of $[\![N]\!]$. Let the color classes of this coloring be $A_1, \ldots, A_t$, for some t. If $|A_i| \geqslant \delta N$ for some i, then A_i contains a k-AP, hence there is a monochromatic k-AP in the coloring. So we may assume that $|A_i| < \delta N$ for all i.

A k-AP is rainbow if all of its elements receive distinct colors. Equivalently, a k-AP is *not* rainbow if it intersects some A_i in at least two points. By part (b), the number of non-rainbow k-APs is thus at most $\sum_{i=1}^{t} \binom{k}{2}|A_i|^2$. Note that

$$\sum_{i=1}^{t} \binom{k}{2}|A_i|^2 = \binom{k}{2}\sum_{i=1}^{t}|A_i|^2 < \binom{k}{2}\sum_{i=1}^{t}(\delta N)|A_i| = \delta\binom{k}{2}N^2 = \frac{N^2}{6k},$$

where we used our assumption $|A_i| < \delta N$ and the fact that we have a coloring of $[\![N]\!]$ to conclude that $\sum|A_i| = N$, and we plug in our choice of δ. By part (a), we conclude that the number of non-rainbow k-APs is strictly less than the total number of k-APs, hence there is at least one rainbow k-AP. $\square$

6. The *Ramsey game* is played between two players, called Builder and Painter. The game starts with an infinite set of vertices and no edges. At every turn, Builder selects a pair of vertices that are not joined by an edge, and builds an edge between them. Painter then immediately has to assign that edge a color, red or blue. The game ends when a monochromatic K_k is produced. Builder's goal is to end the game as soon as possible, while Painter's goal is to continue for as long as possible. The *online Ramsey number*, denoted $\widetilde{r}(k)$, is the minimum number of edges built during the game if both players play optimally.

   (a) Prove that $\widetilde{r}(k) \geqslant \frac{1}{2} \cdot 2^{k/2}$.

   (b) Prove that $\widetilde{r}(k) \leqslant 4^k$.

Solution.

(a) To prove a lower bound $\widetilde{r}(k) \geqslant m$, it suffices to exhibit a strategy for Painter which guarantees that at least m edges are built regardless of Builder's strategy. Set $m = \frac{1}{2} \cdot 2^{k/2}$ and $N = 2m$. The strategy Painter uses is as follows: he first fixes a coloring $\chi : E(K_N) \to \{\text{red}, \text{blue}\}$ with no monochromatic K_k (such a coloring exists by Theorem 2.2.2, since $r(k) > N$). Label the vertices of K_N as $v_1, \dots, v_N$.

Now, when Builder builds the first edge, Painter calls its endpoints v_1, v_2, and colors it according to $\chi(v_1 v_2)$. Every time Builder builds an edge between previously used vertices, Painter again colors it according to χ. Every time Builder builds an edge incident to a new vertex, Painter gives it the next unused label v_i, and again colors all edges according to χ.

Note that if Builder builds at most m edges, then in particular, she touches at most $2m = N$ vertices. Hence, in the stragegy above, the final colored graph produced will be a subgraph of K_N with the coloring χ. As χ has no monochromatic K_k, there will be no monochromatic K_k produced during this process, proving that $\widetilde{r}(k) \geqslant m$.

(b) We now describe a strategy for Builder that is guaranteed to produce a monochromatic K_k after building at most 4^k edges. Builder begins by picking a set S_1 of 2^{2k-1} vertices, and fixing some vertex $v_1 \in S_1$. She then builds all edges from v_1 to the other vertices in S_1. Regardless of how Painter paints these edges, among them, there are at least $\lceil (|S_1| - 1)/2 \rceil = 2^{2k-2}$ of the same color, which we call c_1. Let S_2 be the endpoints of these edges. Builder now repeats the process in S_2: she picks a vertex $v_2 \in S_2$, and builds all edges from v_2 to the rest of S_2. Among these, at least $\lceil (|S_2| - 1)/2 \rceil = 2^{2k-3}$ have the same color, say c_2. Builder restricts to their endpoints S_3, and keeps going.

Note that in this process, we have $|S_i| = 2^{2k-i}$ for all i, hence this process can continue up to the choice of v_{2k}, and we have produced a sequence $c_1, \dots, c_{2k-1}$ of colors. By the pigeonhole principle, there are k of these colors that are the same, yielding a monochromatic K_k.

All that remains is to verify that in this process, Builder builds at most 4^k edges. Indeed, in the first phase, builder builds $|S_1| - 1$ edges, namely all edges from v_1 to the rest of S_1. In the second phase, she builds $|S_2| - 1$ edges, and so on. So the total number of edges built is

$$\sum_{i=1}^{2k}(|S_i| - 1) = -2k + \sum_{i=1}^{2k} 2^{2k-i} = -2k + \sum_{j=0}^{2k-1} 2^j = (2^{2k} - 1) - 2k < 4^k. \qquad \square$$