Minimum degree and the graph removal lemma

Yuval Wigderson (Stanford)

Joint with Jacob Fox

June 30, 2021

Outline

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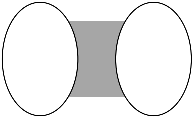

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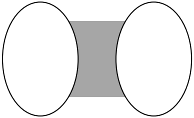

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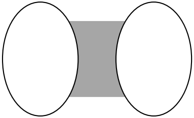

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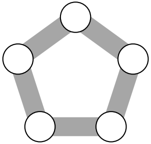

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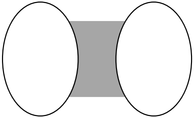

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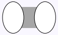
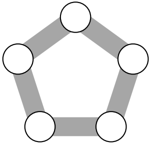
If $\delta(G) > \frac{2}{5}n$, then G is a subgraph of .



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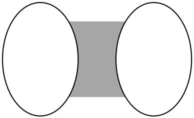

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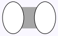
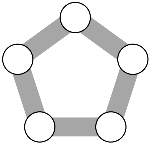
If $\delta(G) > \frac{2}{5}n$, then G is a subgraph of .



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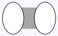
If $\delta(G) > \frac{3}{8}n$, then G is a subgraph of  *or* .

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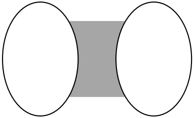

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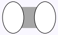
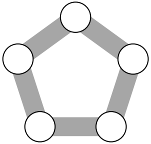
If $\delta(G) > \frac{2}{5}n$, then G is a subgraph of .



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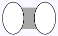
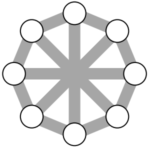
If $\delta(G) > \frac{3}{8}n$, then G is a subgraph of  *or* .



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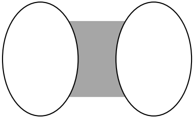

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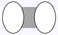
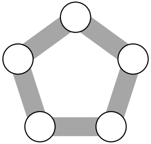
If $\delta(G) > \frac{2}{5}n$, then G is a subgraph of .



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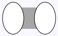
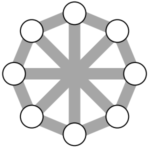
If $\delta(G) > \frac{3}{8}n$, then G is a subgraph of  *or* .



Jin found seven more thresholds for seven more structures.

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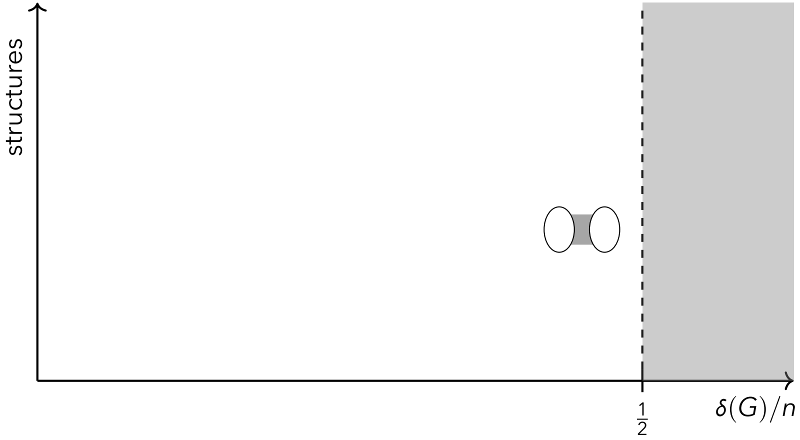

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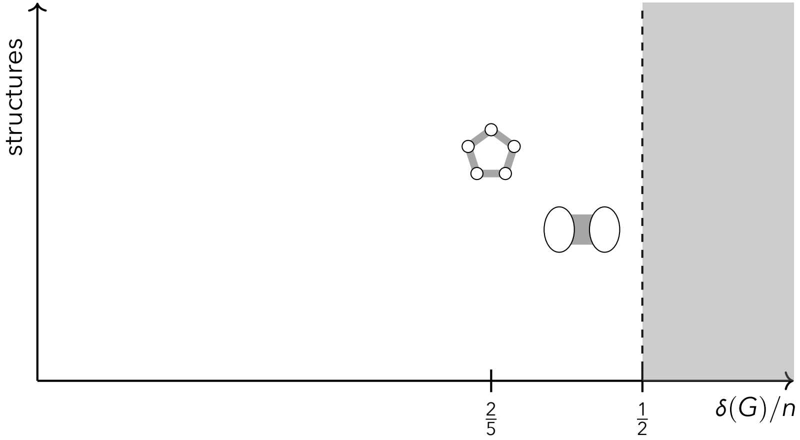

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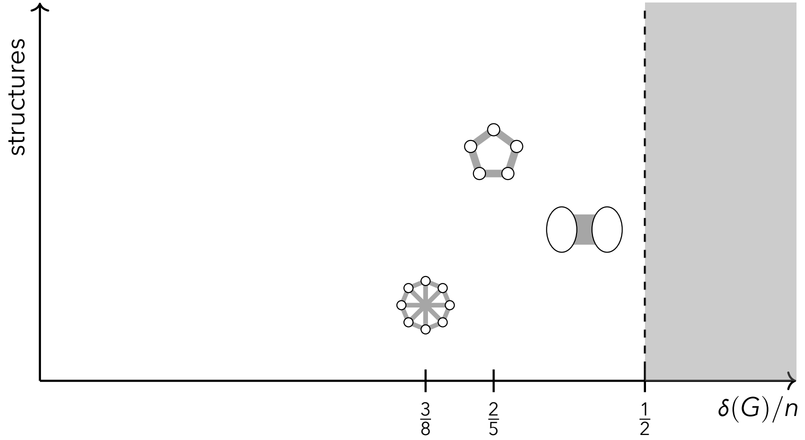

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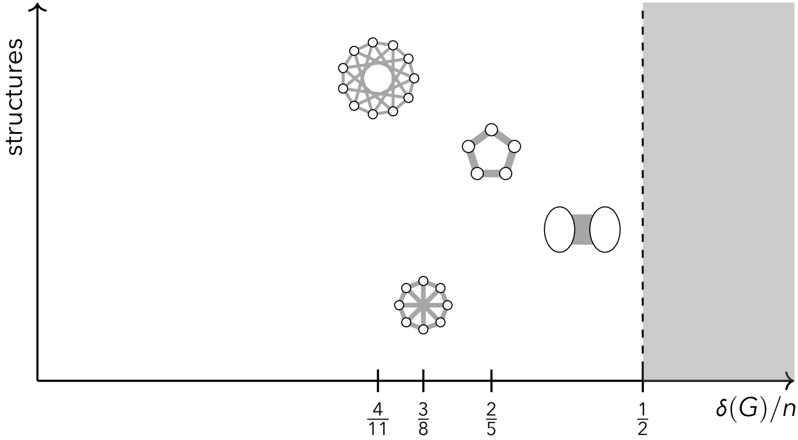

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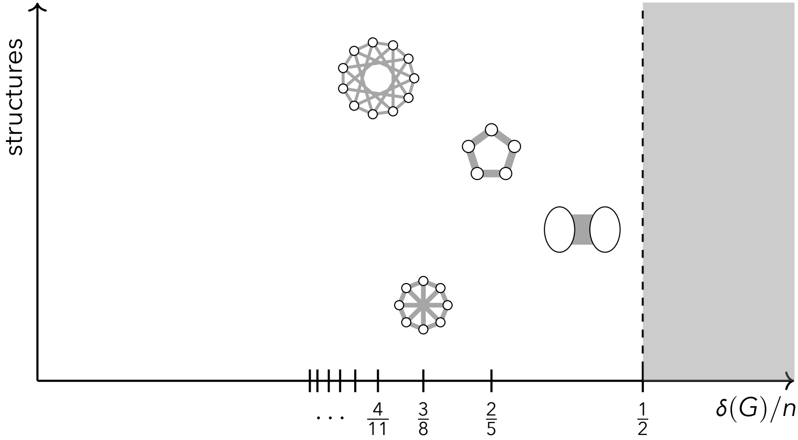

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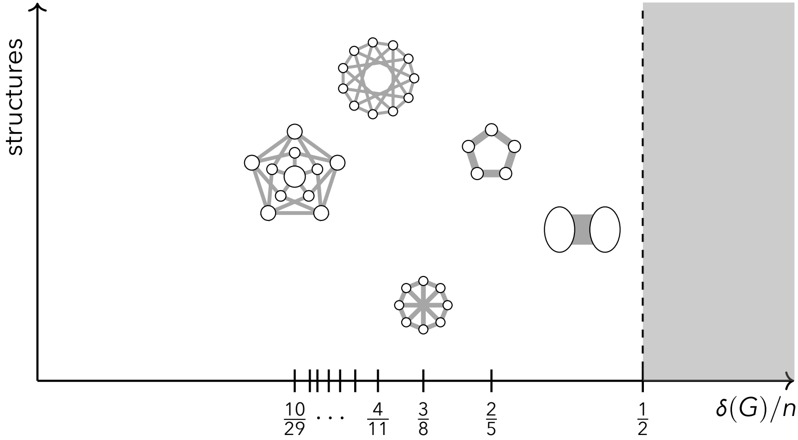

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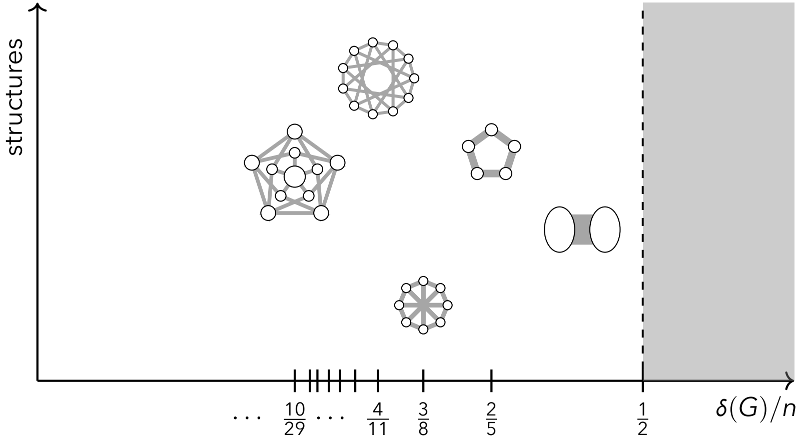

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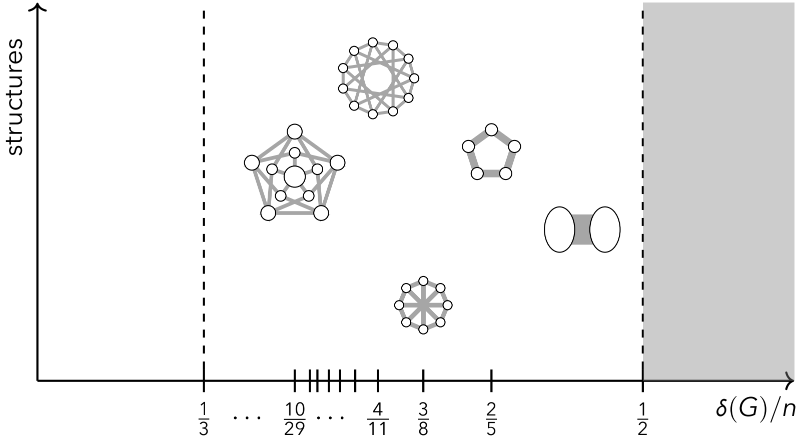

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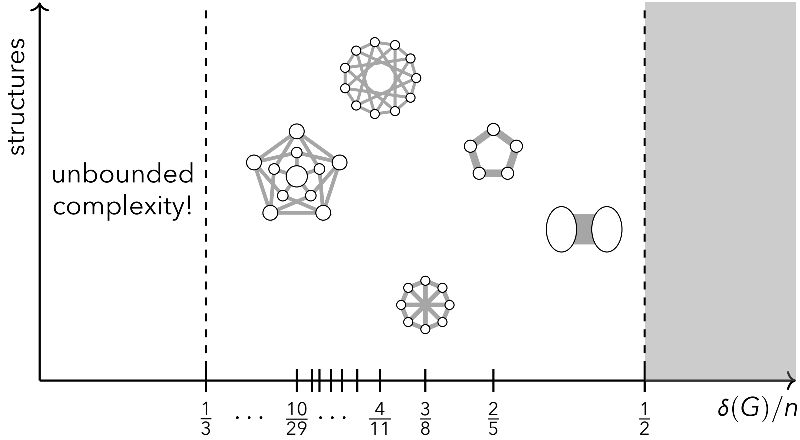

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Graphs with few triangles

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Triangle removal lemma (Ruzsa–Szemerédi)

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Theorem (Alon–Duke–Lefmann–Rödl–Yuster, Füredi)

For every H and $\varepsilon > 0$, there exists $\rho > 0$ such that if G has $< \rho n^{|V(H)|}$ copies of H, then it can be made H-free by removing $< \varepsilon n^2$ edges.

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Original proofs

$$\rho(\varepsilon, H)^{-1} \leq \left. 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \right\} \varepsilon^{-C_H}$$

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Upshot: $\rho(\varepsilon, H)$ is super-polynomial in ε for non-bipartite H.

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Theorem (Fox–W.)

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Triangle removal with high minimum degree

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If $\delta(G) > (\frac{1}{3} + \alpha)n$ and G has $< \rho n^3$ triangles, then G can be made triangle-free by deleting $< \frac{3}{\alpha}\rho n^2$ edges.

In a sufficiently dense graph, the removal lemma has linear bounds!

Proof.

Consider any triangle $\{v_1, v_2, v_3\}$ in G. We have $|N(v_i)| > (\frac{1}{3} + \alpha)n$.
By pigeonhole, $|N(v_i) \cap N(v_j)| > \alpha n$ for some i, j.
So $v_i v_j$ lies in $> \alpha n$ triangles.
So every triangle contains a popular edge, one lying in $> \alpha n$ triangles. Deleting all popular edges destroys all triangles.
t popular edges yield $> \alpha n t/3$ triangles. So $\alpha n t/3 < \rho n^3$, and therefore $t < \frac{3}{\alpha}\rho n^2$. □

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The proof only uses simple averaging arguments!

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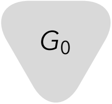

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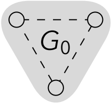

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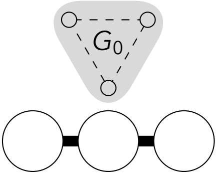

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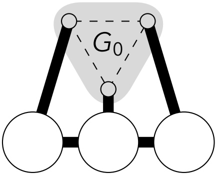

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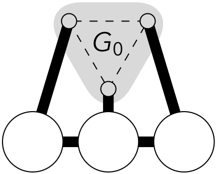

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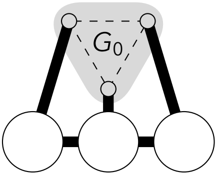

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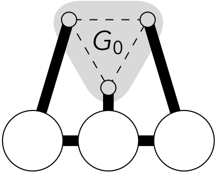

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$\frac{1}{3}$ is a threshold for bounds in the triangle removal lemma.

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Proof sketch: linear bounds for K_r removal

Theorem (Fox–W.)

If $\delta(G) > (\frac{2r-5}{2r-3} + \alpha)n$ and G has $< \rho n^r$ copies of K_r, then G can be made K_r-free by deleting $< c_{\alpha,r}\rho n^2$ edges.

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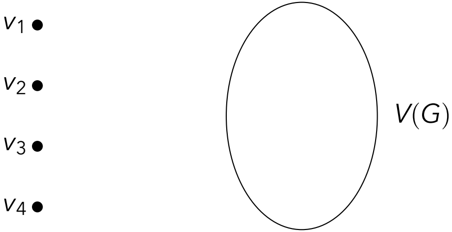

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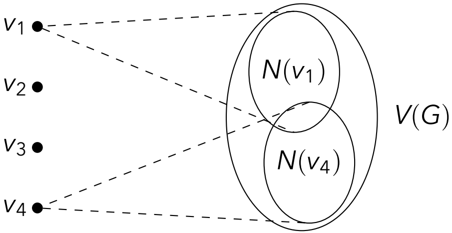

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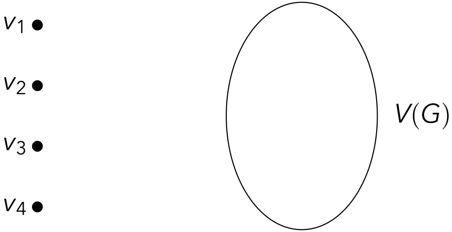

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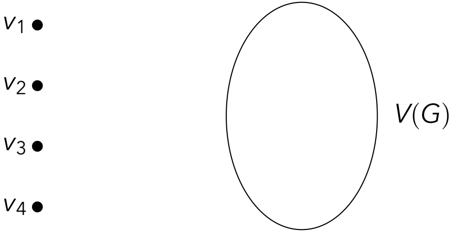

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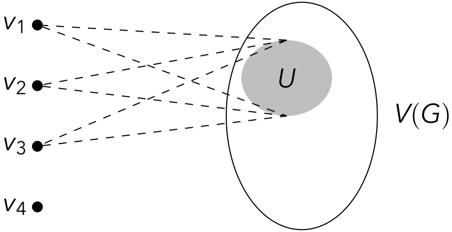

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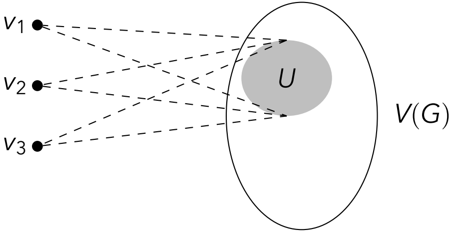

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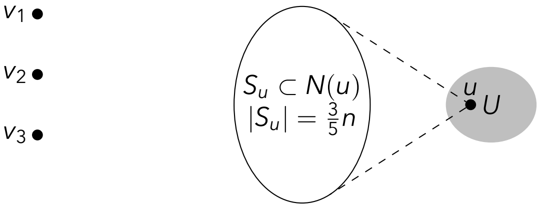

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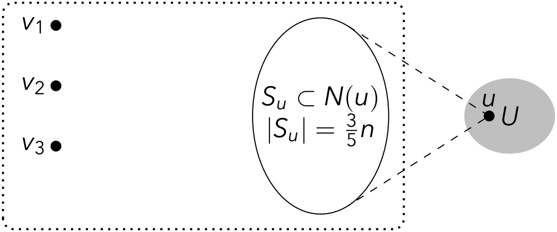

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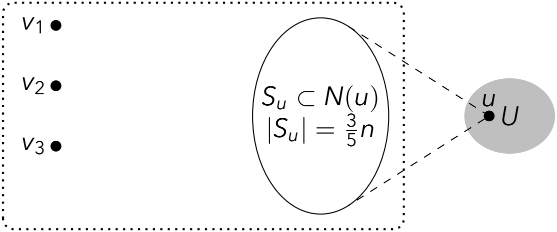

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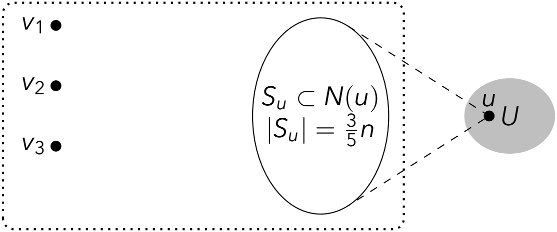

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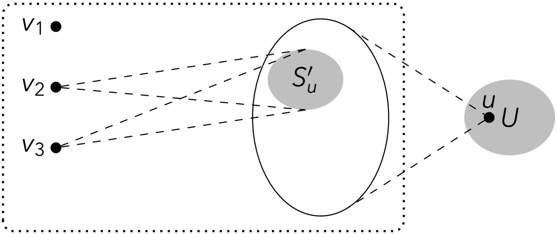

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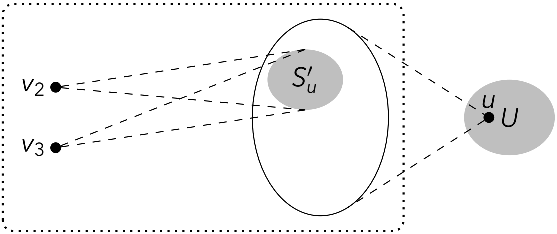

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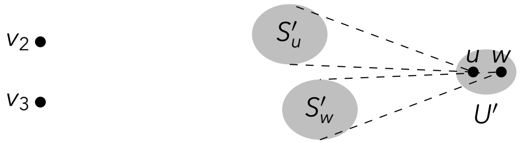

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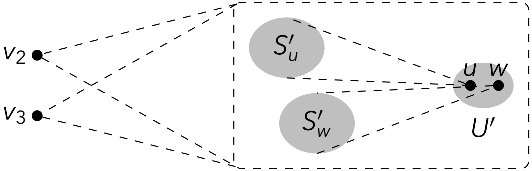

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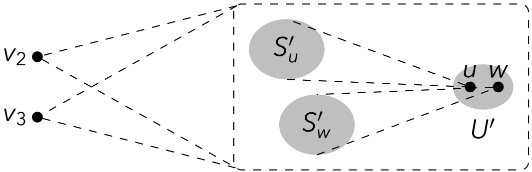

There are $c_\alpha n^2$ edges among the common neighbors of v_2 and v_3.

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- Are there hypergraph analogues of these results?

Thank you!