Three-term arithmetic progressions

Yuval

Mathcamp 2022

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- 3, 5, 7 ($d = 2$)
- 100, 200, 300 ($d = 100$)
- 1, 25, 49 ($d = 24$)
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There are infinitely many 3-APs among the perfect squares.
For any integers r, m, n with $m > n$, let $d = 4r^2mn(m^2 - n^2)$. Then

$$r^2(m^2+n^2)^2 - d, \qquad r^2(m^2+n^2)^2, \qquad r^2(m^2+n^2)^2 + d$$

forms a 3-AP of squares, and all such 3-APs arise in this way.

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Theorem (Dénes 1952, Ribet 1997, Darmon–Merel 1997)

There are no 3-APs among the perfect nth powers, for any $n \geq 3$.

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Theorem (Green–Tao 2008)

The primes contain arbitrarily long arithmetic progressions.

Which sets are 3-AP-free?

Some natural sets of integers (perfect squares, primes) contain infinitely many 3-APs, but others do not.

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How large can a set $A \subseteq \{1, 2, \ldots, N\}$ be if A contains no 3-AP?

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Let $r_3(N)$ be the maximum size of a 3-AP-free subset of $\{1, \ldots, N\}$, i.e.

$$r_3(N) = \max_{\substack{A \subseteq \{1,\ldots,N\} \\ A \text{ contains no 3-AP}}} |A|.$$

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Rest of the talk: How well can we estimate $r_3(N)$ as a function of N?

Thinking asymptotically

Which is larger:

$$X = 1000N^{1/3} + 1000000 \qquad \text{or} \qquad Y = \frac{N^{2/3}}{1000} - 1000000 \quad ?$$

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For two functions f, g, we say that $f \gtrsim g$ if

$$\lim_{N \to \infty} \frac{f(N)}{g(N)} > 0.$$

$f \gtrsim g$ means that f is at least as large as g up to a constant factor.

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Lower bounds on $r_3(N)$

Techniques for lower-bounding $r_3(N)$

Recall: $r_3(N)$ is the maximum size of a 3-AP-free subset of $\{1, 2, \ldots, N\}$.

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Probabilistic method

Greedy algorithm

Number theory

High-dimensional geometry

The probabilistic method

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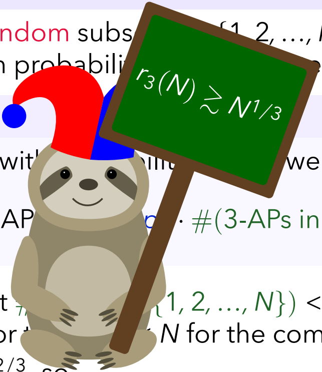

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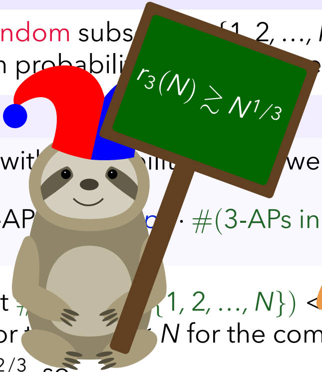

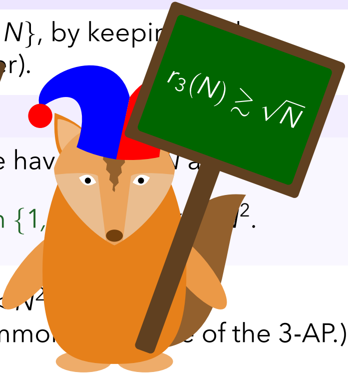

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They rule out at most $3\binom{k}{2}$ other elements.

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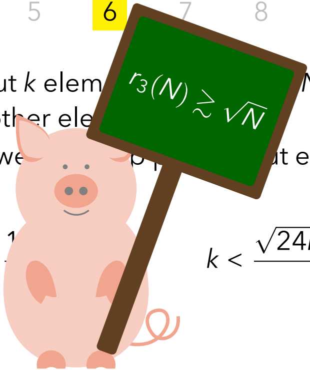

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Suppose for simplicity $N = 3^m - 1$, so every number in $\{1, \ldots, N\}$ has m base-3 digits (with leading 0s).

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Suppose for simplicity $N = 3^m - 1$, so every number in $\{1, \ldots, N\}$ has m base-3 digits (with leading 0s).

$$|T| = 2^m - 1$$

A number-theoretic construction

Idea (Szekeres)

Consider all numbers with no 2 in their base-3 expansion.

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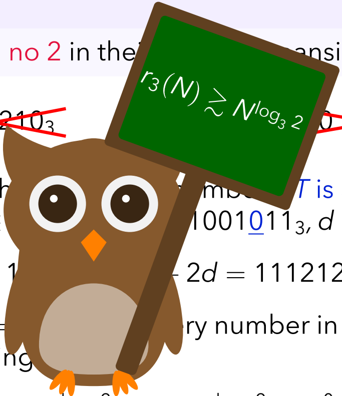

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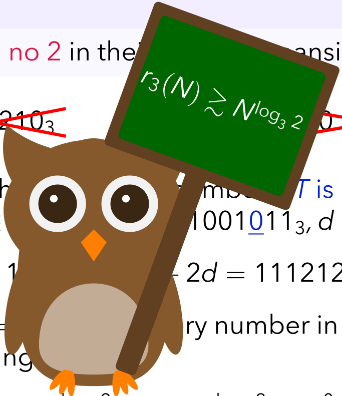

Conjecture (Erdős–Turán 1936)

This is best possible, i.e. $r_3(N) \lesssim N^{\log_3 2}$.

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Idea (Behrend 1946)

A 3-AP is 3 points, one of which is the midpoint of the other two.

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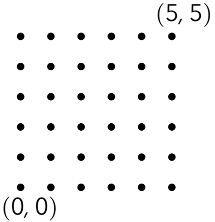

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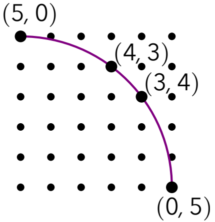

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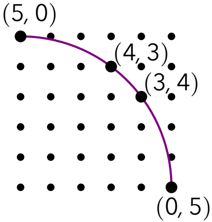

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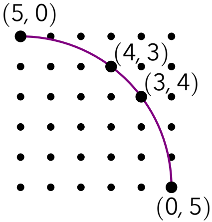

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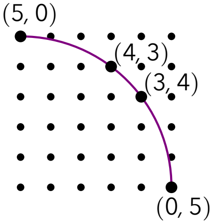

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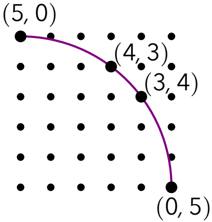

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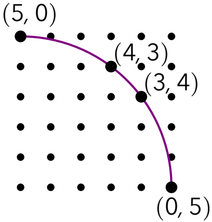

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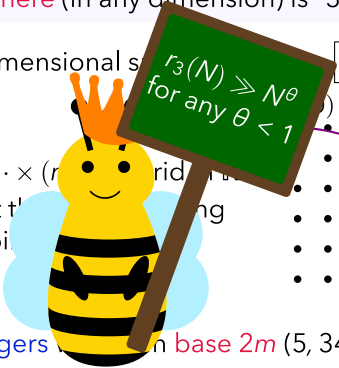

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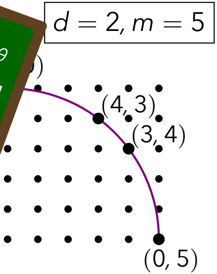

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Upper bounds on $r_3(N)$

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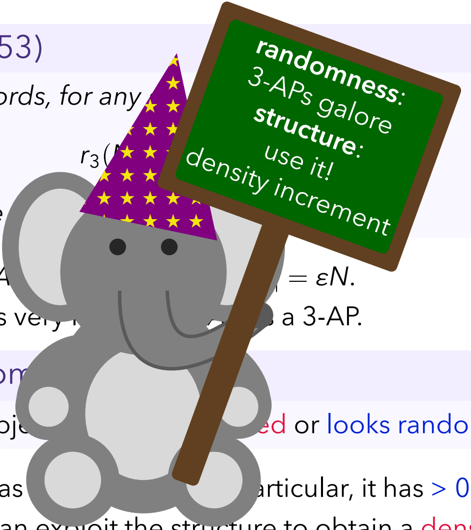

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Corollary

The primes contain infinitely many 3-APs—not because the primes are special, just because there are a lot of them!

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$$r_k(N) \ll N \text{ for all } k.$$


Thank you!