1 Recall that given a graph G, its *complement graph* $\overline{G}$ is the graph whose vertices are the same as those of G, but whose edges are precisely the complement of the edges of G—in other words, if (u, v) is an edge of G, then (u, v) is not an edge of $\overline{G}$, and vice versa.

(a) Let C_n be a cycle graph of length n, and let P_n be a path graph on n vertices (i.e. C_n minus an edge). Draw the complement graphs $\overline{C_4}, \overline{C_5}, \overline{C_6}, \overline{P_3}, \overline{P_4}, \overline{P_5}$.

(b) Verify that $\overline{C_5}$ is isomorphic to C_5 and that $\overline{P_4}$ is isomorphic to P_4. Graphs with this property (that $\overline{G} \cong G$) are called *self-complementary.*

(c) Using the isomorphism between C_5 and $\overline{C_5}$, reconstruct the example I gave in class of five length-2 strings that are not confusable from C_5.

(d) Generalize the construction above to prove that if G is a self-complementary graph with n vertices, then $\Theta(G) \geq \sqrt{n}$.

(e)* Try to find more self-complementary graphs!

2 In this problem, we'll explore a very important quantity associated to a graph, called its *fractional chromatic number.*

(a) Let $\mathcal{I}(G)$ denote the collection of all non-empty independent sets in a graph G. Write down all elements of $\mathcal{I}(C_5)$ (make sure you write down 10 independent sets!).

(b) A *fractional coloring* of G is an assignment of a real number $a_I \geq 0$ for each $I \in \mathcal{I}(G)$ such that for every vertex v of G,

$$\sum_{\substack{I \in \mathcal{I}(G) \\ v \in I}} a_I \geq 1$$

Recall that a *proper coloring* of G is just a choice of some independent sets in G that cover all the vertices; these chosen independent sets are called *color classes.* Check that if we declare

$$a_I = \begin{cases} 1 & \text{if } I \text{ is a color class} \\ 0 & \text{otherwise} \end{cases}$$

then this yields a fractional coloring. Thus, fractional coloring is a more general notion than proper coloring.

(c) The *fractional chromatic number* of G is defined as

$$\chi_f(G) = \min \left\{ \sum_{I \in \mathcal{I}(G)} a_I \,\middle|\, \{a_I\}_{I \in \mathcal{I}(G)} \text{ is a fractional coloring of } G \right\}$$

Using (2b), check that $\chi_f(G) \leq \chi(G)$.

(d) Find a fractional coloring $\{a_I\}_{I \in \mathcal{I}(C_5)}$ of C_5 such that $\sum_{I \in \mathcal{I}(G)} a_I = \frac{5}{2}$, and thus conclude that $\chi_f(C_5) \leq \frac{5}{2}$. Tomorrow, we will see that the fractional chromatic number is an upper bound for the Shannon capacity (i.e. $\Theta(G) \leq \chi_f(G)$ for all graphs G), so this implies that $\Theta(C_5) \leq \frac{5}{2}$, which is better than our current upper bound of $\chi(C_5) = 3$.

(e)* Try to prove that $\chi_f(C_5) = \frac{5}{2}$.

1 A sequence of real numbers $a_1, a_2, \ldots$ is called superadditive if for every m, n, we have that

$$a_{m+n} \geq a_m + a_n$$

A very useful theorem from analysis is the following, called *Fekete's Lemma.*

Lemma. *If $a_1, a_2, \ldots$ is a superadditive sequence, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists.*

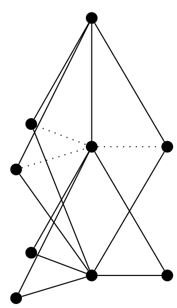
Check that if G is a graph and if we define

$$a_n = \log \omega(G^n)$$

then $a_1, a_2, \ldots$ is a superadditive sequence. Then apply Fekete's Lemma to conclude that the Shannon capacity is well defined.

2 We know that for any graph, $\omega(G) \leq \chi(G)$. We also know that for the five-cycle C_5, we have that $\omega(C_5) = \chi(C_5) - 1$. A natural question is how large the gap between $\omega(G)$ and $\chi(G)$ can be. In this problem, we'll explore the *Mycielski construction*, which shows that this gap can be arbitrarily large.

Given a graph G, its Mycielskian $M(G)$ is a new graph defined as follows. If the vertices of G are $u_1, \ldots, u_n$, then the vertices of $M(G)$ are $u_1, \ldots, u_n, v_1, \ldots, v_n, x$. Among the u_is, we keep connectivity exactly as it was in G, namely if u_i, u_j are adjacent in G, then they're also adjacent in $M(G)$. Between the v_is we don't connect anything, although we connect v_i to u_j if and only if (u_i, u_j) is an edge of G; in other words, each v_i is a "clone" of u_i, and is connected to the u_js that u_i is connected to. Finally, the extra vertex x is connected to all the v_is. Here is a maybe-helpful schematic:



The bottom layer are the u_is (namely just a copy of G), the middle layer are the v_is (with dotted lines not representing edges, just demonstrating the relationships copied from the bottom layer), and the top layer is x.

(a) Check that if $P_2 =$ •——•, then $M(P_2) = C_5$.

(b) Prove that if G is triangle-free (i.e. there is no set of three vertices all pairs of which are adjacent), then $M(G)$ is triangle-free as well.

(c) Prove that $\chi(M(G)) = \chi(G) + 1$.

(d) We define the Mycielski graphs as $M_1 = P_2$, $M_2 = C_5 = M(M_1)$, and more generally $M_n = M(M_{n-1})$. Observe that each M_n is triangle-free, so $\omega(M_n) = 2$ for all n, and that $\chi(M_n) = n+1$. Therefore, these graphs have arbitrarily large gaps between ω and χ. Thus, for these graphs, we get arbitrarily bad bounds on Θ.

3* If you did the homework problem on the fractional chromatic number yesterday, check that it satisfies the two properties in the key lemma we proved today, namely that for any graphs G, H, we have

- $\chi_f(G) \geq \omega(G)$
- $\chi_f(G \cdot H) \leq \chi_f(G) \cdot \chi_f(H)$

Conclude that $\Theta(G) \leq \chi_f(G)$, and thus conclude that $\Theta(C_5) \leq \frac{5}{2}$.

4* In this problem, we will prove Fekete's Lemma (from Problem 1). Let $a_1, a_2, \ldots$ be any superadditive sequence of real numbers.

(a) Pick any $m \geq 1$ and any $n \geq m$. Divide n by m with remainder to write $n = qm + r$, where q, r are integers and $0 \leq r \leq m-1$. By applying the superadditivity property many times, prove that

$$a_n \geq qa_m + a_r$$

(b) Divide by n to get

$$\frac{a_n}{n} \geq \frac{q}{n}a_m + \frac{a_r}{n} \geq \frac{q}{n}a_m + \frac{\min_{0 \leq \ell \leq m-1} a_\ell}{n}$$

Now, let $n \to \infty$ while keeping m fixed. Prove that the right-hand side of the above inequality converges to a_m/m.

(c) Conclude that the sequence $a_1/1, a_2/2, \ldots, a_m/m, \ldots$ can't oscillate between a high and a low value infinitely many times, and thus that $\lim_{m \to \infty} a_m/m$ exists.

1 This problem will be about the geometry of regular pentagons; tomorrow, we will see that the facts here will end up being very useful for us.

(a) Prove that if a regular pentagon has side length 1, then the length of any diagonal is the golden ratio,

$$\phi = \frac{1+\sqrt{5}}{2}$$

(b) Prove that if a regular pentagon has side length 1, then the distance from any vertex to the center is

$$\sqrt{\frac{1}{2}+\frac{\sqrt{5}}{10}} = \frac{\sqrt{\phi}}{\sqrt[4]{5}}$$

(c) Check that

$$\phi = \phi\sqrt{5} - 2$$

2* For any $k \geq 2$, a *k-vector coloring* of a graph G is an assignment of vectors $\mathbf{u}_i \in \mathbb{R}^n$, one for each vertex of G, such that if i and j are adjacent vertices, then

$$\mathbf{u}_i \cdot \mathbf{u}_j = -\frac{1}{k-1}$$

(a) Prove that if we place three points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^2$ so that they form an equilateral triangle centered at the origin, then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_3 = \mathbf{v}_1 \cdot \mathbf{v}_3 = -\frac{1}{2}$$

(b) Similarly, prove that if we place four points in $\mathbb{R}^3$ so that they form a regular tetrahedron centered at the origin, then the dot product of any pair of them will be equal to $-1/3$.

(c)* More generally, prove that $n+1$ points in $\mathbb{R}^n$ that form a regular "simplex" centered at the origin have the property that the dot product of any pair equals $-1/n$.

(d) Prove that if k is an integer and we have a proper k-coloring of a graph G, then we can also get a k-vertex coloring of G.
Hint: Place each color class at a vertex of a regular simplex in $\mathbb{R}^{k-1}$.

(e) The *vertex chromatic number* of G is defined as

$$\chi_v(G) = \min\{k : \text{there exists a } k\text{-vector coloring of } G\}$$

Note that we allow k to be any real number, not just an integer. Using the previous part, prove that $\chi_v(G) \leq \chi(G)$.

(f)* Try to prove that $\chi_v(G) = \vartheta(G)$.

1 To solve this problem, you'll need some familiarity with modular arithmetic and a little bit of number theory.

If p is a prime that is 1 mod 4, then we define the *Paley graph* of order p, denoted P_p, as follows. The vertices are just the integers mod p, namely $0, 1, \ldots, p-1$. Two distinct vertices are adjacent in this graph if their difference is a perfect square mod p.

(a) Prove that this is a well-defined graph, namely that if x is adjacent to y, then also that y is adjacent to x (notice that this is *not* immediate from the definition). To prove this, you may assume a standard fact from number theory, namely that if $p \equiv 1 \mod 4$, then -1 is a perfect square mod p.

(b) Check that the Paley graph of order 5 is just C_5.

(c) Prove that all Paley graphs are self-complementary (and thus solve a homework problem from the first assignment).
Hint: To construct an isomorphism between P_p and $\overline{P_p}$, fix a non-perfect-square z mod p, and map every vertex x to xz.

(d) Conclude that $\Theta(P_p) \geq \sqrt{p}$, using a problem on the first homework assignment.

(e)* Try to prove that $\vartheta(P_p) \leq \sqrt{p}$, which implies that $\Theta(P_p) = \sqrt{p}$.
It is worth remarking that the clique number and chromatic number of P_p are unknown in general; it is believed that $\omega(P_p) \approx \log p$ and $\chi(P_p) \approx p/\log p$ for most values of p, although it seems that proving anything like this will involve a major breakthrough in number theory (e.g. proving the Riemann Hypothesis). Thus, Paley graphs are in sharp contrast to the graphs we've been studying for much of this course (e.g. odd cycles), for which we know the clique and chromatic numbers, but have to work really hard to determine the Shannon capacity.

2* We've almost finished determining the Shannon capacity of C_5. However, the Shannon capacity of any larger odd cycle is still unknown (and new bounds, even if they are a very small improvement, would likely be considered a major breakthrough). In this problem, we'll look a bit at the best known results for these larger odd cycles.

(a) It turns out that if k is odd, then

$$\vartheta(C_k) = \frac{1 + \cos(\frac{\pi}{k})}{\cos(\frac{\pi}{k})}$$

and this is our best-known upper bound on $\Theta(C_k)$. For the case $k = 7$, try to prove this, or at least prove that

$$\vartheta(C_7) \leq \frac{1 + \cos(\frac{\pi}{7})}{\cos(\frac{\pi}{7})}$$

I believe that you can do this by exhibiting a suitable 4-dimensional orthonormal representation of C_7, but I'm not sure; I can't picture $\mathbb{R}^4$ well enough.

(b) As for lower bounds, the best known result is the following, due to Bohman and Holzman:

$$\Theta(C_{2n+3}) \geq \left(2^{2^n} + 1\right)^{1/2^n} > 2$$

Note that though this beats the clique-number lower bound of 2, it does so *very* slightly. They proved this result by exhibiting a clique of size $2^{2^n} + 1$ in the OR power $C_{2n+3}^{2^n}$. Try to do this for the 7-cycle, namely the case $n = 2$: try to find a clique of size 17 in C_7^4.

(c)* Can you improve on either of these bounds?