Recent results on size Ramsey numbers

David Conlon

SIAM Conference on Discrete Mathematics

June 14, 2022

Joint with Jacob Fox and Yuval Wigderson

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Observation (Erdős–Faudree–Rousseau–Schelp)

If H has no isolated vertices, then

$$\frac{r(H)}{2} \leq \hat{r}(H) \leq \binom{r(H)}{2}.$$

Linear or quadratic?

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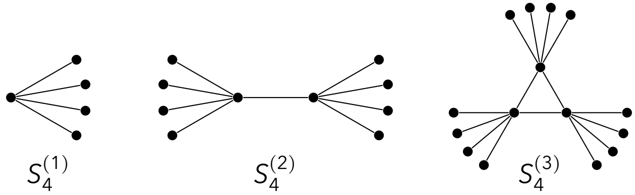

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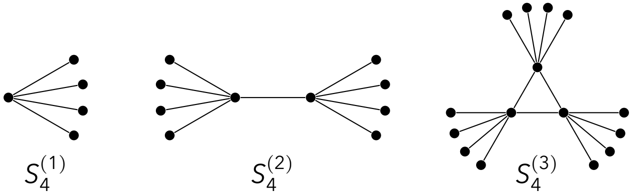

Every n-vertex tree T has $r(T) = \Theta(n)$, so $r(S_n^{(1)}), r(S_n^{(2)}) = \Theta(n)$.

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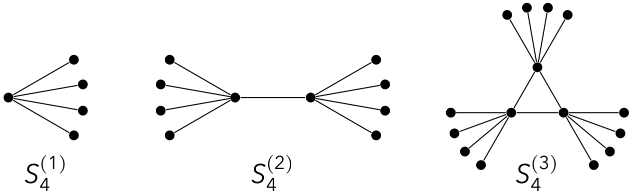

Every n-vertex tree T has $r(T) = \Theta(n)$, so $r(S_n^{(1)}), r(S_n^{(2)}) = \Theta(n)$.
However:

Proposition

$\hat{r}(S_n^{(1)}) = 2n - 1$. *On the other hand,* $\hat{r}(S_n^{(2)}) = \Theta(n^2)$.

Complete graphs

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Proof.

Exercise!

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Hint: Prove that if G is Ramsey for K_t, then $\chi(G) \geq r(K_t)$. □

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Theorem (Erdős, Erdős–Szekeres)

$$2^{t/2} \leq r(K_t) \leq 2^{2t}$$

The four questions

Problem (Erdős–Faudree–Rousseau–Schelp)

Determine the size Ramsey numbers of the following graph families.

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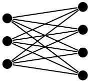

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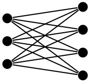

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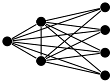

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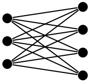

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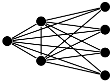

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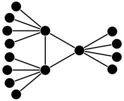

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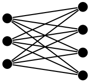

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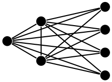

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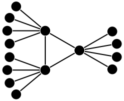
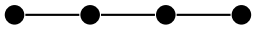


Path graphs, P_n

P_4



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Any good expander G on $\Theta(n)$ vertices is Ramsey for P_n.
In particular, we may take $G = G(N, p)$ for $N = \Theta(n)$, $p = \Theta(\frac{1}{n})$. □

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Often, one takes G to be a sparse (pseudo)random graph. To prove that G is Ramsey for H, one uses techniques like sparse regularity.

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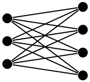

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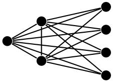

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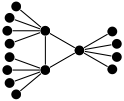

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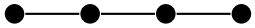

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Lemma (Erdős–Rousseau)

For any $s \le t$ *and any graph* G,

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By the lemma, this is < 1, so there exists a coloring with no monochromatic $K_{s,t}$.

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Pikhurko proved that if $t \gg s$, then this construction (appropriately optimized) is optimal. In particular, $\hat{r}(K_{s,t}) = \Theta(s^2 t 2^s)$ if $t \gg s$.

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So rather than uniform, it's better to use a (dyadically iterated) hypergeometric random coloring.

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Problem (Erdős–Faudree–Rousseau–Schelp)

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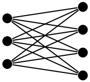
$K_{3,4}$



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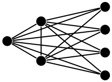

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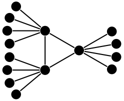

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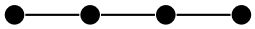

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These are the only obstructions to finding monochromatic books.
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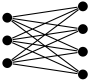
$K_{3,4}$ 

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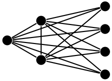
$B_4^{(3)}$ 

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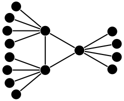
$S_4^{(3)}$ 

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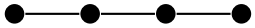
P_4 

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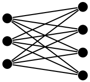

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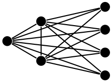

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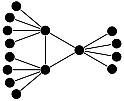

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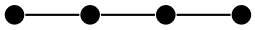

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