The Four Cube-Face Fun Fact of "The Barn"

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Fun Fact (The Barn, 1770)**.** *Each num that is more than zero can be made as a sum of four or less of a cube-face.*

We will show this Fun Fact with a list of many a tiny step.

1 Many a fact on many an atom

Tiny Fun Fact 1. *If the "Fun Fact of The Barn" is true for each atom, then it is true for each num more than zero.*

How to Show. If we can get $n = a^2+b^2+c^2+d^2$ and $m = w^2+x^2+y^2+z^2$ as the sum of four of a cube-face, we can get n by m in the same way, like this:

$$n \cdot m = (a^2+b^2+c^2+d^2)(w^2+x^2+y^2+z^2) = (aw-bx-cy-dz)^2 + (ax+bw+cz-dy)^2 + (ay-bz+cw+dx)^2 + (az+by-cx+dw)^2 .$$

(Do it out to see that it is true.) We can get to each num if we do this with one atom at a time, so if the Fun Fact is true for atoms, we are done. ☐

How did math men of yore find this long fact? Some say that it does come from the norm of nums in two-bar H: the norm of $q_1 = a+bi+cj+dk$ by $q_2 = w+xi+yj+zk$ is the same as the norm of q_1 by the norm of q_2. But it is more true to say that two-bar H does come from this Fun Fact, as this is what made men of the past want to know more that two-bar H can tell us.

Tiny Fun Fact 2. *If p is an odd atom, the nums $0^2, 1^2, 2^2, \dots, m^2$ (when m is half of one less than p) are all not the same mod p.*

In fact, each cube-face is the same as one of $0^2, 1^2, \dots, m^2$ mod p, but we do not need this fact.

How to Show. Say we had a and b with $0 \le a, b \le m$ and $a^2 \equiv b^2 \bmod p$. Then we can say $0 \equiv a^2 - b^2 \equiv (a-b)(a+b)$. We know that $a+b \not\equiv 0$, for $0 < a+b < 2m < p$. From this, we must have that $a-b \equiv 0 \bmod p$, so that $a \equiv b \bmod p$. ☐

Tiny Fun Fact 3. *For each atom p, we can find two nums α and β such that 1, α by α, and β by β add to a num that p goes into: $p \mid 1 + a^2 + b^2 \equiv 0$.*

How to Show. If $p = 2$, then take $\alpha = 1$ and $\beta = 0$. So now say that p is odd. By Tiny Fun Fact 2, we know that the sets

$$A := \{\alpha^2 \mid 0 \le \alpha \le m\}$$
$$B := \{-1-\beta^2 \mid 0 \le \beta \le m\}$$

both have size $m+1$. Since $m = \frac{p-1}{2}$, we have that $m+1 = \frac{p+1}{2}$. This does tell us that

$$|A| + |B| = \frac{p+1}{2} + \frac{p+1}{2} = p+1 > p$$

so by the Bird-Hole Fun Fact, we must have a num that is in both A and B mod p: $\alpha^2 \equiv -1-\beta^2$. That is to say, $\alpha^2+\beta^2+1 \equiv 0 \bmod p$. ☐

2 Many a grid

Term 1. To form a *grid* in d dims, pick d of a dart that are a base of $\mathbb{R}^d$, and look at sums of a set of them and what you get when you flip them. You can use each more than once. Each spot you land on is part of the grid, and that is it.

If you take each dart in the base and put it as a row to form a cube-face of nums, the det of this is what we mean by the *d-dim area of the grid.*

Note that each grid may come from many a base, but each base does give the same d-dim area (try to show this!).

Fun Fact (Mink Man)**.** *Let G be a d-dim grid with d-dim area A. Then an open ball with d-dim area more than $2^d A$ with a mid-dot at zero must also have in it a grid dot that is not zero.*

How to Show. Say that grid G does come from base $v_1, \ldots, v_d$, so each spot in $\mathbb{R}^d$ has the form $c_1v_1+\cdots+c_dv_d$ for real nums c_i. Look at the map that does send such a spot to

$$(c_1 \bmod 2) \cdot v_1 + \cdots + (c_d \bmod 2) \cdot v_d,$$

that lies in the skew box that you form with each base dart as half an edge. E.g., the spot $4v_1 + 6.3v_2 + \pi v_3$ does go to the spot $.3v_2 + (\pi - 2)v_3$. Note that this skew box has d-dim area $\det(2v_1, \ldots, 2v_d) = 2^d \cdot A$.

What does this map do to the ball? It cuts the ball into many a hunk, and then it does move each hunk into the skew box. The area of any hunk does not go up or down, but to fit all of them into the skew box (which has less area than the ball), some hunk must hit one more hunk. That is to say, this map must send two dots w_1 and w_2 in the ball to the same spot in the skew box. So, $w_2 - w_1$ is a dart that does look like $2n_1v_1 + \cdots 2n_dv_d$ with each n_i an int. So, the spot half way from w_2 to the flip of w_1 (i.e., $-w_1$), is the spot $n_1v_1 + \cdots n_dv_d$ that lies in the grid G. As w_2 and $-w_1$ both lie in the ball, so does this grid point. $\square$

3 Do it all at once

How to show the Fun Fact of The Barn. By Tiny Fun Fact 1, we know that we only need to show The Fun Fact of The Barn for nums that are an atom. So let p be an atom, and by Tiny Fun Fact 3, we can find α and β such that $\alpha^2 + \beta^2 + 1 \equiv 0 \bmod p$. Form the 4-dim grid G from this base:

$$\begin{aligned}
&(1, 0, \alpha, \beta)\\
&(0, 1, \beta, -\alpha)\\
&(0, 0, p, 0)\\
&(0, 0, 0, p).
\end{aligned}$$

This is the same as to say that a list (w, x, y, z) of ints is in G iff

$$\begin{aligned}
\alpha w + \beta x &\equiv y \bmod p\\
\beta w - \alpha x &\equiv z \bmod p.
\end{aligned}$$

If we look at the cube-face of nums from the base, we find that this grid has 4-dim area of p^2. Also, for each spot (w, x, y, z) in G, if we take the norm to the 2, we get an int that p goes into:

$$\begin{aligned}
w^2 + x^2 + y^2 + z^2 &\equiv w^2 + x^2 + (\alpha w + \beta x)^2 + (\beta w - \alpha x)^2\\
&\equiv w^2 + x^2 + \alpha^2 w^2 + 2\alpha\beta wx + \beta^2 x^2 + \beta^2 w^2 - 2\alpha\beta wx + \alpha^2 x^2\\
&\equiv (w^2 + x^2)(1 + \alpha^2 + \beta^2)\\
&\equiv 0.
\end{aligned}$$

Now, look at the open ball with mid-dot at $(0, 0, 0, 0)$ that goes just up to $(r, 0, 0, 0)$ with $r = \sqrt{2p}$ (but does not have this spot in it). You can look up that the 4-dim area of this ball is $\frac{\pi^2}{2} r^4 = 2\pi^2 p^2$. As $2\pi^2 > 16 = 2^4$, we can use the Fun Fact of Mink Man and say that this ball has a grid spot (w_0, x_0, y_0, z_0) that is not the same as $(0, 0, 0, 0)$. We know that $0 < w_0^2 + x_0^2 + y_0^2 + z_0^2 < r^2 = 2p$ and also that $p \mid w_0^2 + x_0^2 + y_0^2 + z_0^2$, so it must be true that $w_0^2 + x_0^2 + y_0^2 + z_0^2 = p$. Done! $\square$

Key that lets you know what a word does mean

atom	prime
base	basis
cube-face	square
cube-face of nums	matrix
d-dim area	volume
dart	vector
det	determinant
grid	lattice
Mink Man	Minkowski
The Barn	Lagrange
Two-bar H	$\mathbb{H}$, i.e., the Quaternions
skew box	parallelepiped