Finding structures in tournaments

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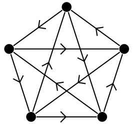
ETH Zürich

2ª Escola Brasileira de Combinatória
March 13, 2025

Hamiltonian paths in tournaments

A tournament is a complete directed graph (every pair of vertices is connected by a directed edge).



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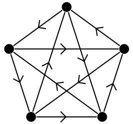

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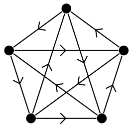

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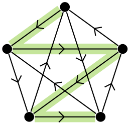

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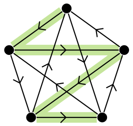

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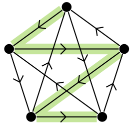

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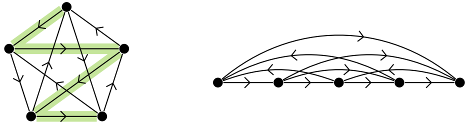

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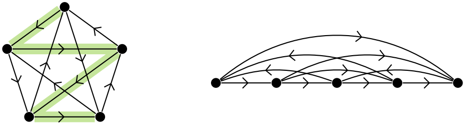

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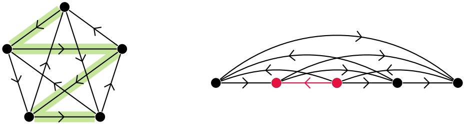

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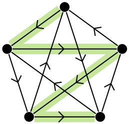

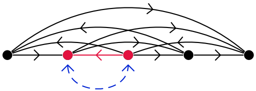

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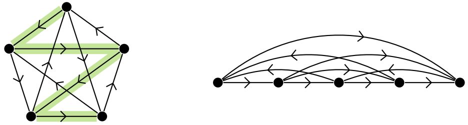

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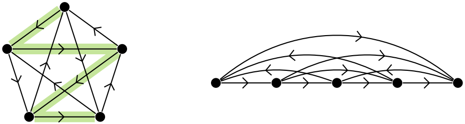

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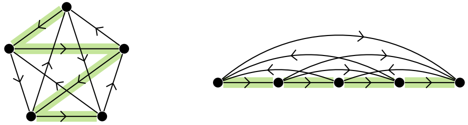

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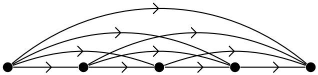

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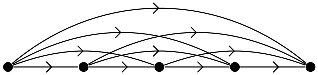

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The only structures we can hope to find in every tournament are acyclic.

Some structures we can find

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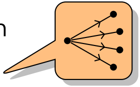
out-directed star 		$\lceil \frac{N+1}{2} \rceil$

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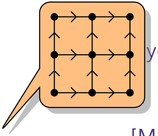
up-right oriented grid	[Morawski–W. '24+]	$N/10^{12}$



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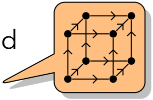

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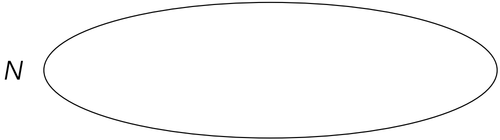

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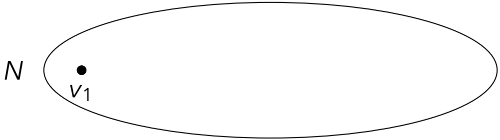

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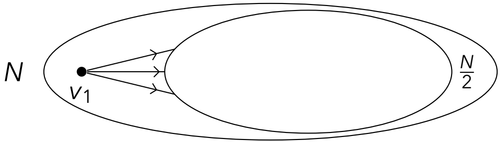

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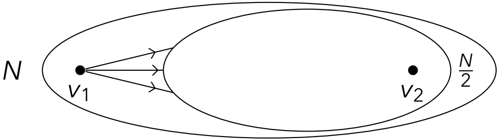

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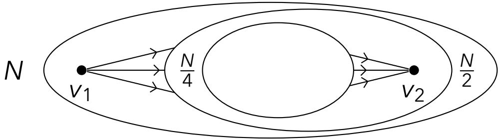

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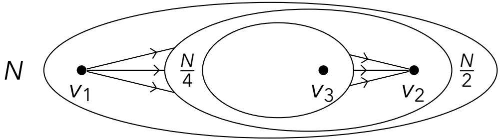

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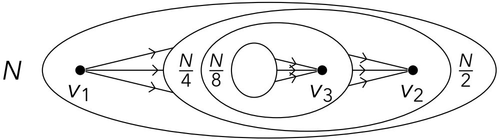

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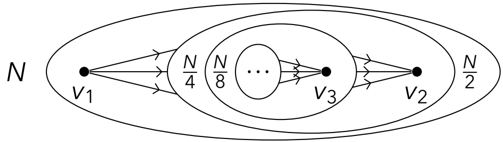

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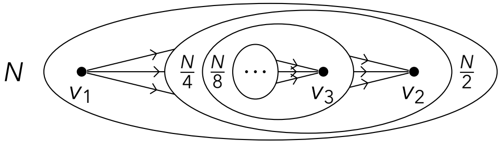

Theorem (Erdős–Moser 1964)

There exists an N-vertex tournament with no transitive subtournament on $2 \log N$ vertices.

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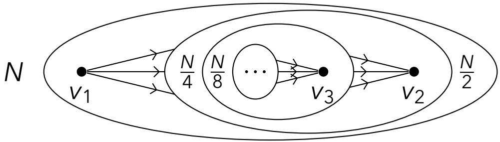

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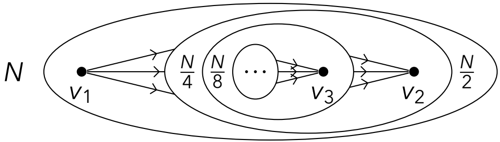

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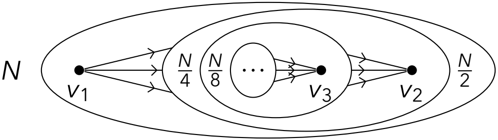

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For $k = 2 \log N$, this quantity is < 1. □

Ramsey numbers of digraphs

Question: What structures exist in every N-vertex tournament?

Every N-vertex tournament has...	on ... vertices
directed path	N
any oriented path	N
out-directed star	$\lceil \frac{N+1}{2} \rceil$
any oriented tree	$\lceil \frac{N+1}{2} \rceil$
any* oriented cycle	N
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Definition: The Ramsey number $\vec{r}(H)$ of a digraph H is the least N such that every N-vertex tournament contains a copy of H.

Ramsey numbers of digraphs

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If H has n vertices, Every N-vertex tournament has…	on … vertices	$\vec{r}(H) \leqslant \cdots$
directed path	N	n
any oriented path	N	n
out-directed star	$\lceil \frac{N+1}{2} \rceil$	$2n-2$
any oriented tree	$\lceil \frac{N+1}{2} \rceil$	$2n-2$
any* oriented cycle	N	n
up-right oriented grid	$N/10^{12}$	$10^{12}n$
oriented hypercube	$N^{0.244}$	$n^{4.09}$
transitive subtournament	$\log N$	2^n

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$$2^{n/2} \leqslant \vec{r}(\overrightarrow{T_n}) \leqslant 2^n.$$

The upper bound implies that $\vec{r}(H)$ exists for all acyclic H.
If H has εn^2 edges, then

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Definition

The Ramsey number $r(H)$ of a graph H is the minimum N such that every two-edge-coloring of K_N contains a monochromatic copy of H.

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So the Ramsey number is exponential if H is dense.

For the rest of the talk, we'll focus on sparse (di)graphs.

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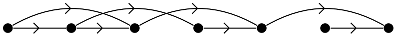

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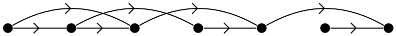

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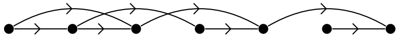

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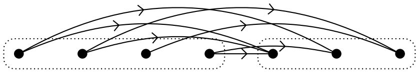

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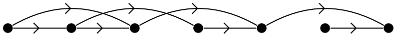

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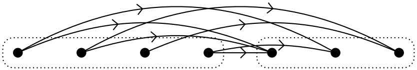

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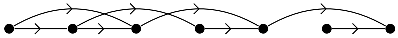

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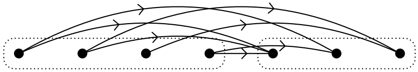

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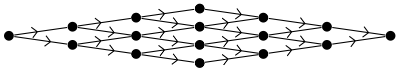

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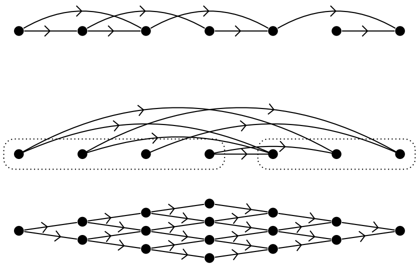

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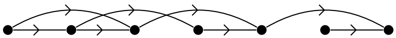

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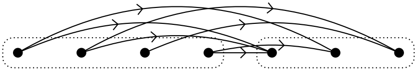

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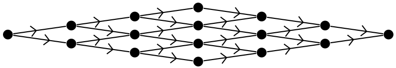

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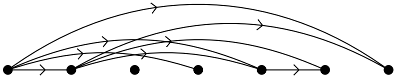

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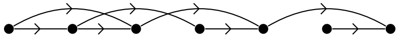

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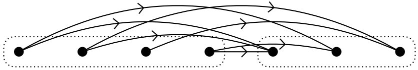

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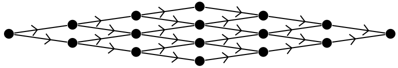

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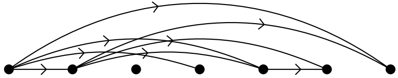

Proofs use many different techniques!

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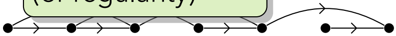

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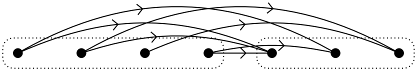

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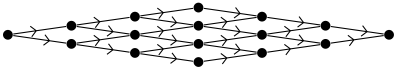

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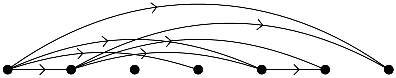

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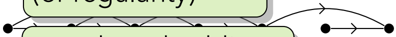

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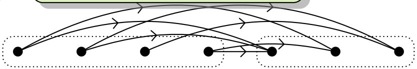

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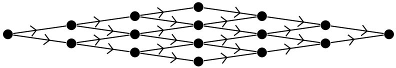

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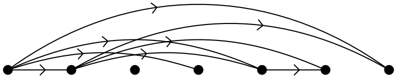

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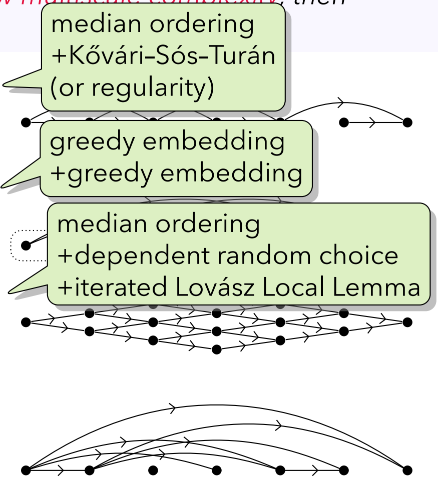

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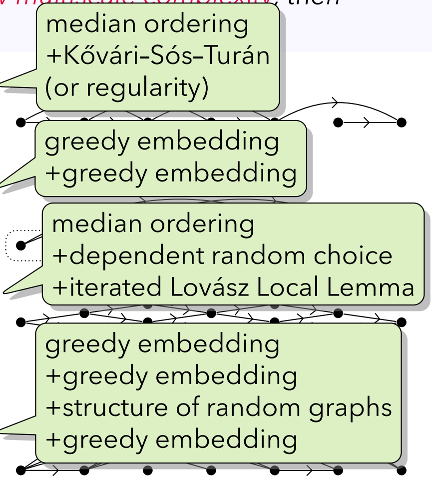

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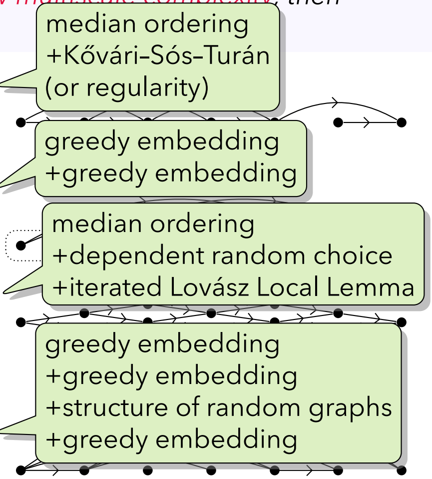

Proofs use many different techniques! Is there a unified argument?

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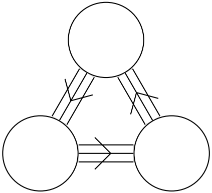

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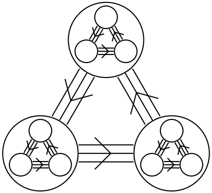

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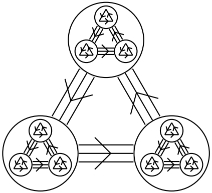

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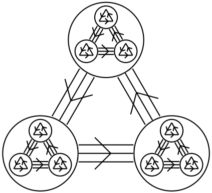

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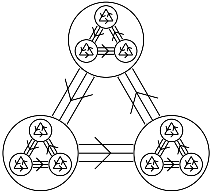

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Ensure that the induced subgraph on this subinterval has the same property, so we can iterate.

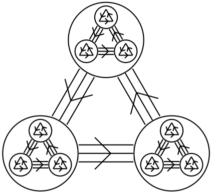
Lower bound proof sketch

Theorem (Fox–He–W. 2024)

There exists an n-vertex acyclic digraph H with maximum degree $\leqslant 1000$ *and* $\vec{r}(H) > n^{\log_2(3)-\varepsilon}$.

We need (1) a construction of H, (2) a tournament T on $n^{\log_2(3)-\varepsilon}$ vertices, and (3) a proof that there is no embedding $H \hookrightarrow T$.

For (2): We let T be an iterated blowup of a cyclic triangle.



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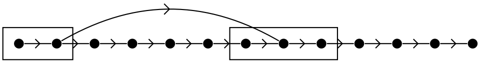

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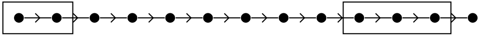

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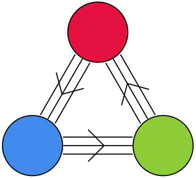

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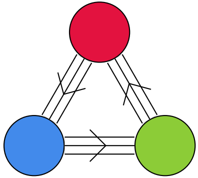

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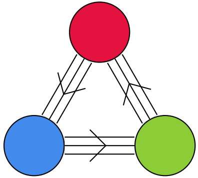

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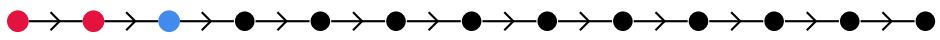

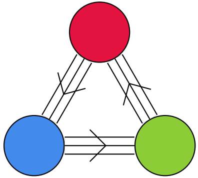

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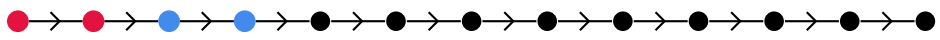

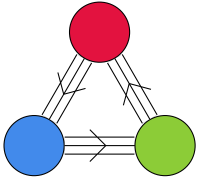

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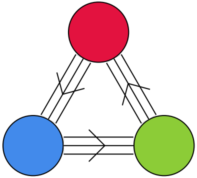

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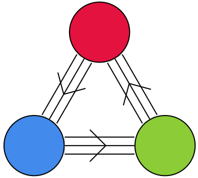

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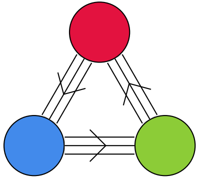

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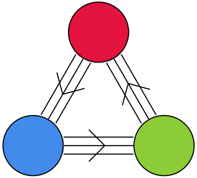

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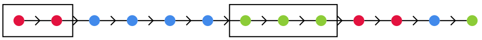

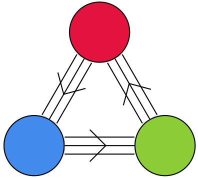

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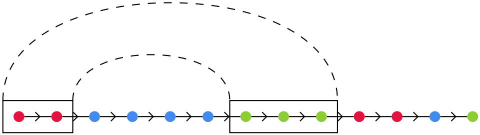

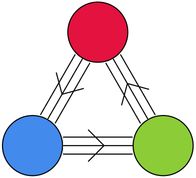

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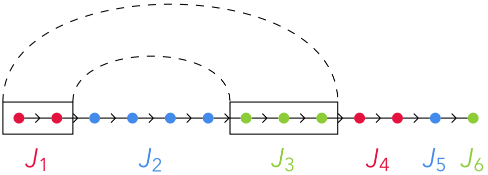

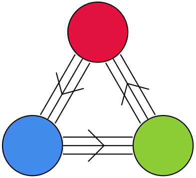

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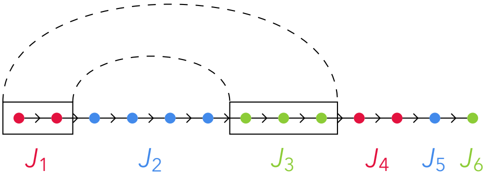

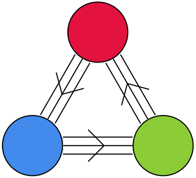

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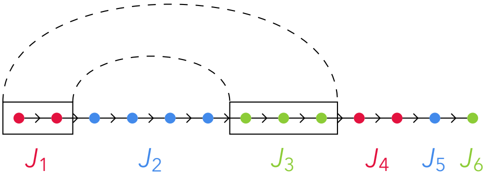

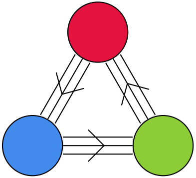

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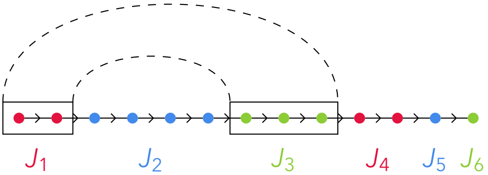

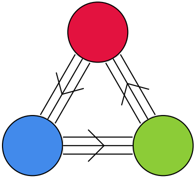

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Greedy algorithm yields an interval mesh with max degree $\leqslant 1000$.

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Theorem (Sudakov 2011)

If H is a graph with m edges, then $r(H) \leqslant 2^{O(\sqrt{m})}$.

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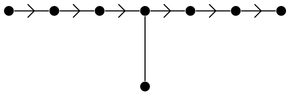

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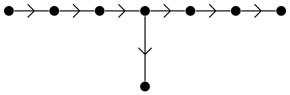

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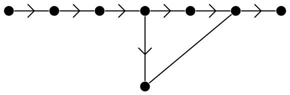

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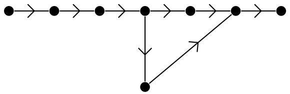

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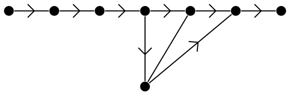

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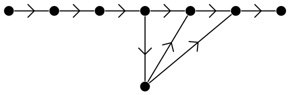

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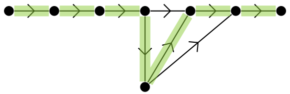

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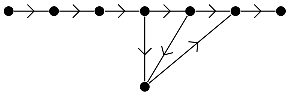

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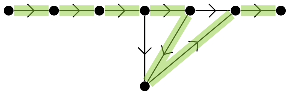

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- Take your favorite result in Ramsey theory, and prove (or disprove!) a directed version of it.

Thank you!