Finding structures in tournaments

Yuval Wigderson

ETH Zürich

Sparse (Graphs) Coalition
Topics in Ramsey theory

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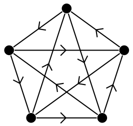
Hamiltonian paths in tournaments

A tournament is a complete directed graph (every pair of vertices is connected by a directed edge).



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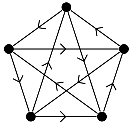

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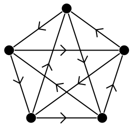

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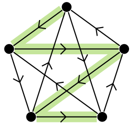

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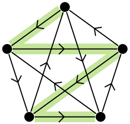

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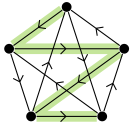

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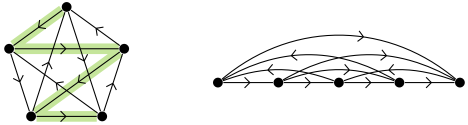

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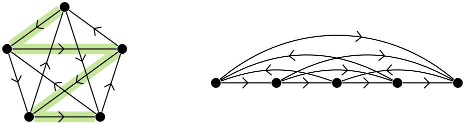

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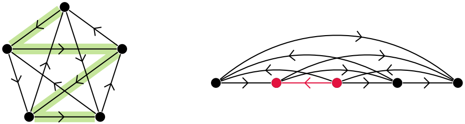

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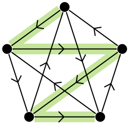

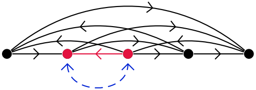

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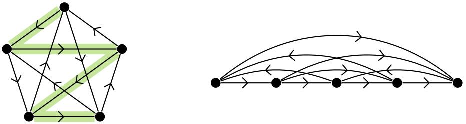

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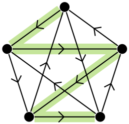

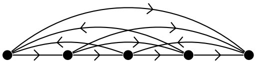

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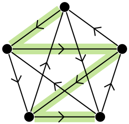

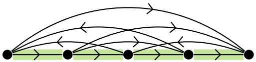

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Does every tournament contain a Hamiltonian directed cycle?

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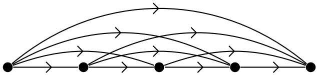

A transitive tournament has no directed cycles at all.

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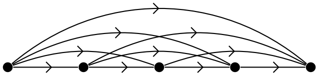

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The only structures we can hope to find in every tournament are acyclic.

Some structures we can find

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directed path [Rédei '34]	N

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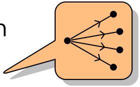
out-directed star 		$\lceil \frac{N+1}{2} \rceil$

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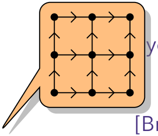
up-right oriented grid	[Bradač–Morawski–Sudakov–W. '25+]	$10^{-12}N$



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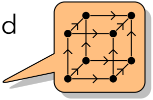

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transitive subtournament	[Stearns '59]	$\log N$

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Theorem (Stearns 1959)

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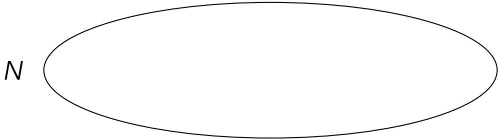

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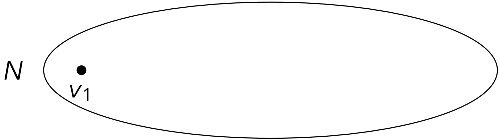

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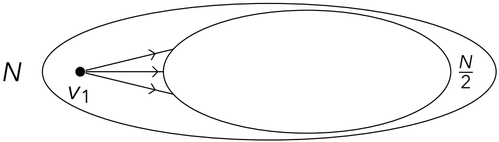

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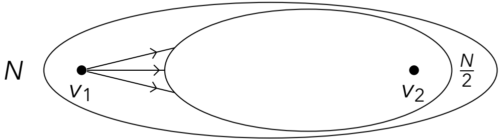

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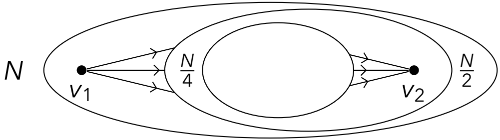

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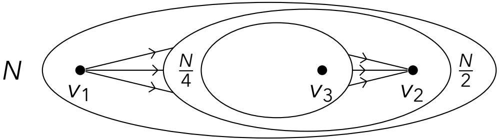

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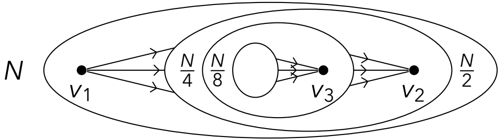

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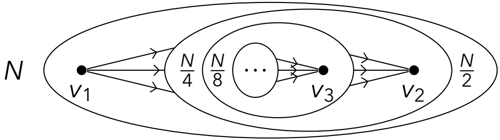

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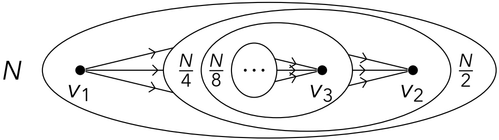

Theorem (Erdős–Moser 1964)

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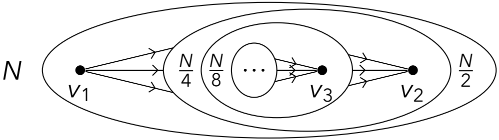

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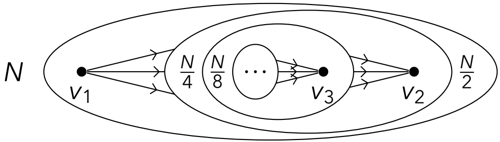

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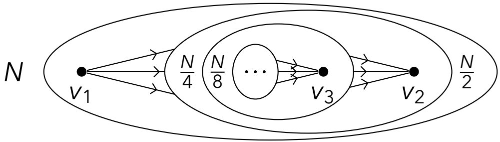

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For $k = 2\log N$, this quantity is < 1. □

Ramsey numbers of digraphs

Question: What structures exist in every N-vertex tournament?

Every N-vertex tournament has…	on … vertices
directed path	N
any oriented path	N
out-directed star	$\lceil \frac{N+1}{2} \rceil$
any oriented tree	$\lceil \frac{N+1}{2} \rceil$
any* oriented cycle	N
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Definition: The Ramsey number $\vec{r}(H)$ of a digraph H is the least N such that every N-vertex tournament contains a copy of H.

Ramsey numbers of digraphs

Question: What structures exist in every N-vertex tournament?

If H has n vertices, Every N-vertex tournament has...	on ... vertices	$\vec{r}(H) \leqslant \cdots$
directed path	N	n
any oriented path	N	n
out-directed star	$\lceil \frac{N+1}{2} \rceil$	$2n-2$
any oriented tree	$\lceil \frac{N+1}{2} \rceil$	$2n-2$
any* oriented cycle	N	n
up-right oriented grid	$10^{-12}N$	$10^{12}n$
oriented hypercube	$N^{0.244}$	$n^{4.09}$
transitive subtournament	$\log N$	2^n

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The upper bound implies that $r(H)$ exists for all H.

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The Ramsey number $r(H)$ of a graph H is the minimum N such that every two-edge-coloring of K_N contains a monochromatic copy of H.

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Directed and undirected Ramsey numbers

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So the Ramsey number is exponential if H is dense.

For the rest of the talk, we'll focus on sparse (di)graphs.

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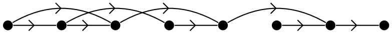

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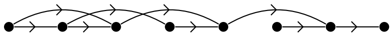

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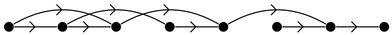

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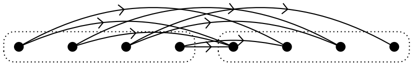

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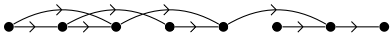

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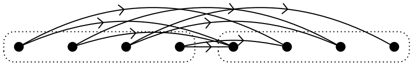

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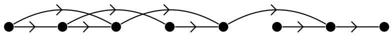

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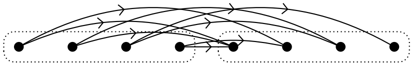

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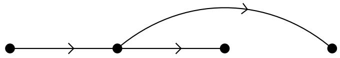

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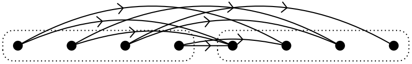

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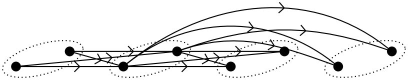

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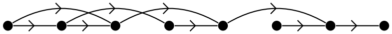

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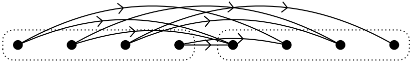

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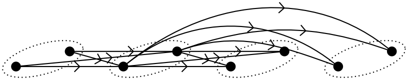

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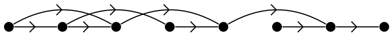

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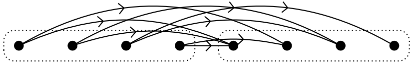

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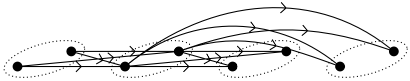

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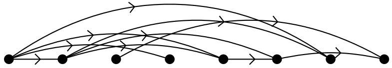

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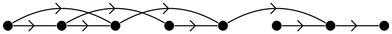

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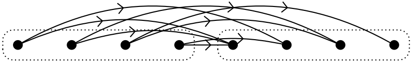

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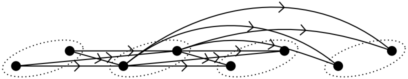

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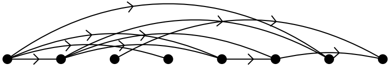

Proofs use many different techniques!

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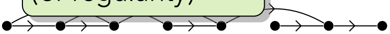

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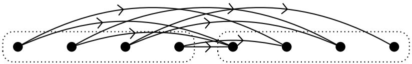

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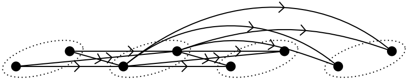

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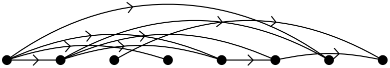

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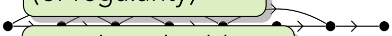

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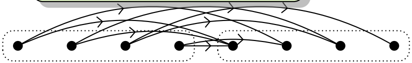

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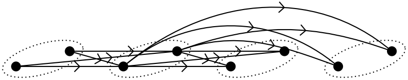

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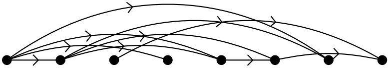

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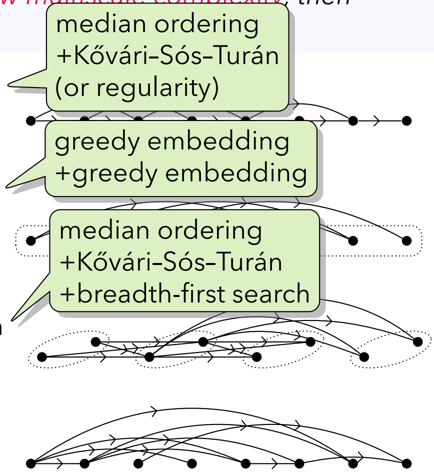

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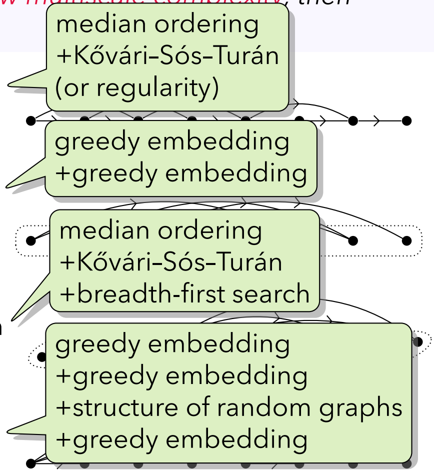

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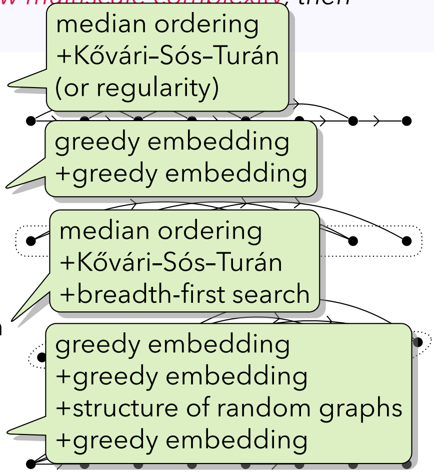

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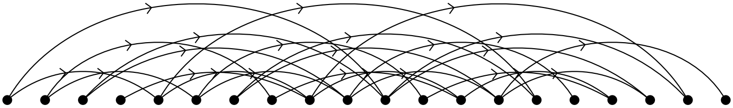

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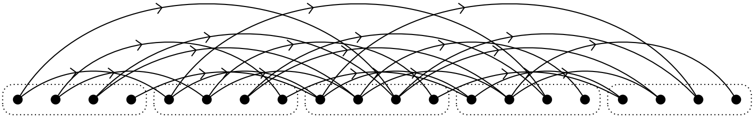

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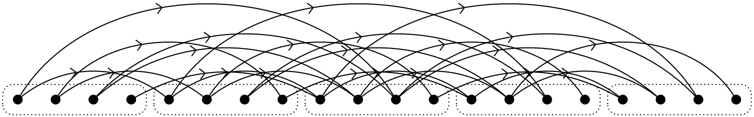
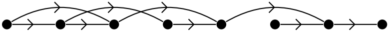


Bounded bandwidth 

A unified statement

Theorem (Bradač–Morawski–Sudakov–W. 2025+)

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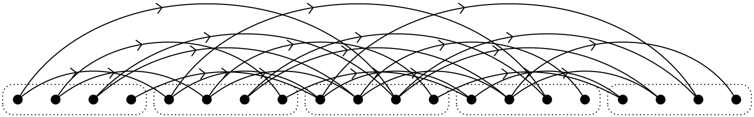

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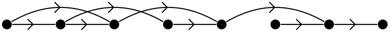

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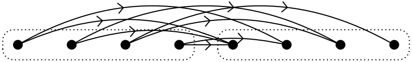

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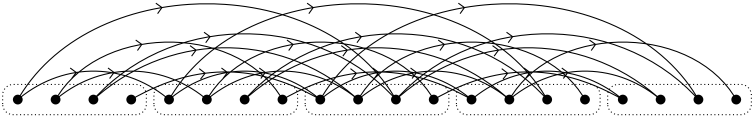

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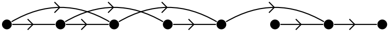

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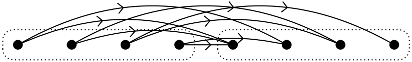

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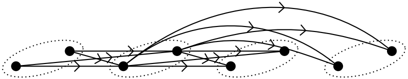

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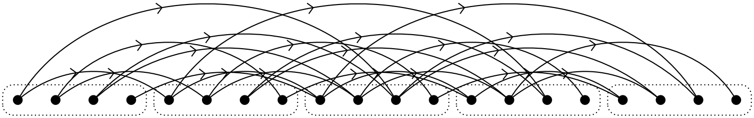

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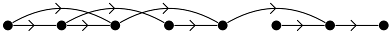

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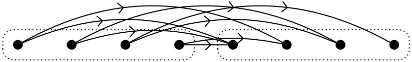

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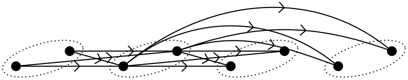

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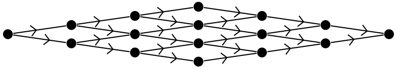

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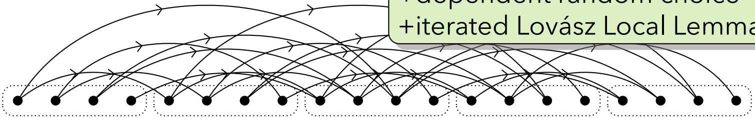
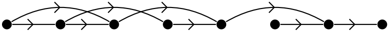
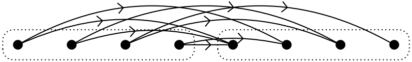
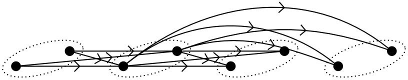
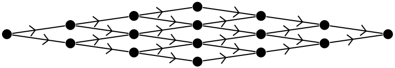


Bounded bandwidth 

Bounded height 

Bounded blowup of a tree 

Graded 

Lower bound proof sketch

Lower bound proof sketch

Theorem (Fox–He–W. 2024)

For all $C > 0$, there exists a bounded-degree n-vertex acyclic digraph H with $\vec{r}(H) > n^C$.

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Theorem (Fox–He–W. 2024)

There exists an n-vertex acyclic digraph H with maximum degree $\leqslant 1000$ and $\vec{r}(H) > n^{\log_2(3)-\varepsilon}$.

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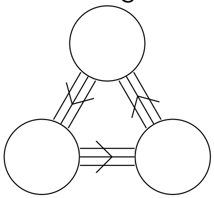

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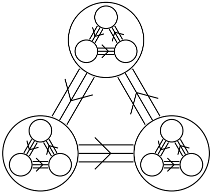

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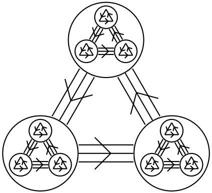

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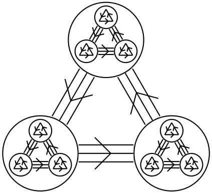

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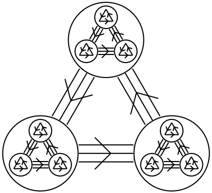

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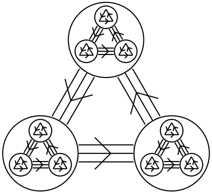

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Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $\geqslant 0.49n$ is mapped into a single part, and this is hereditary.

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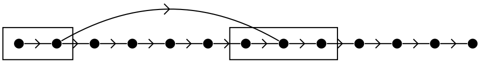

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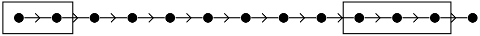

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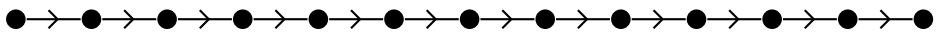

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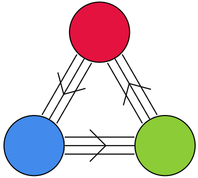

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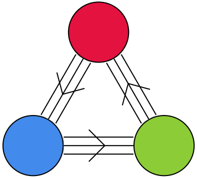

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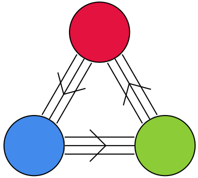

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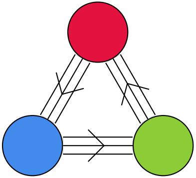

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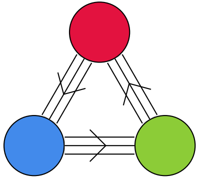

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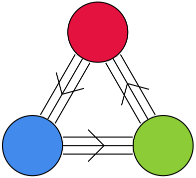

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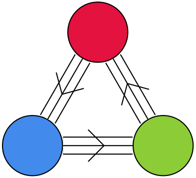

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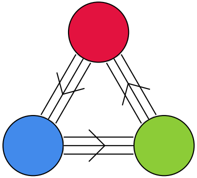

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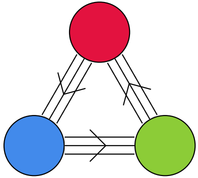

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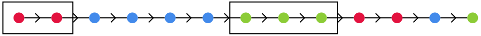

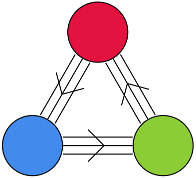

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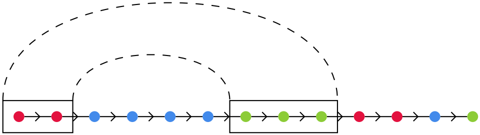

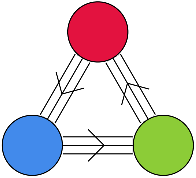

Lower bound proof sketch: interval meshes

Want: In any embedding $H \hookrightarrow T$, some subinterval of $[n]$ of length $\geqslant 0.49n$ is mapped into a single part, and this is hereditary.

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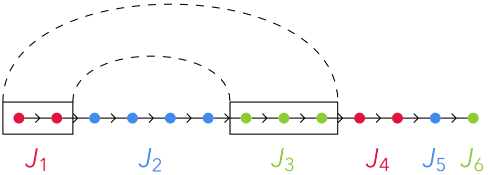

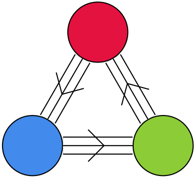

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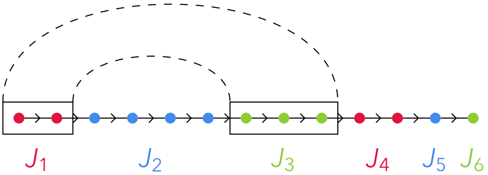

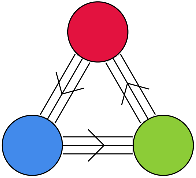

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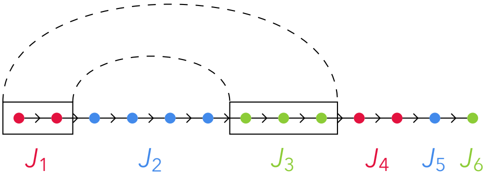

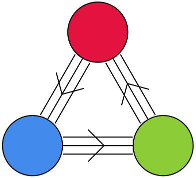

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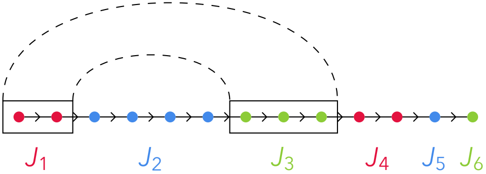

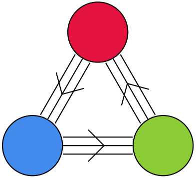

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Greedy algorithm yields an interval mesh with max degree $\leqslant 1000$.

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If H is a graph with m edges, then $r(H) \leqslant 2^{O(\sqrt{m})}$.

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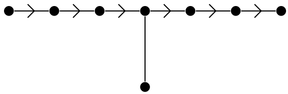

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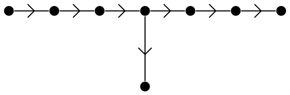

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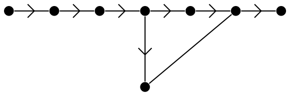

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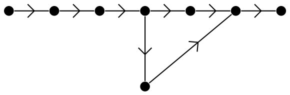

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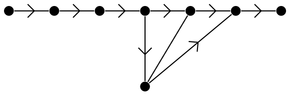

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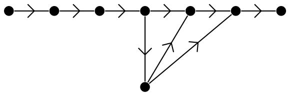

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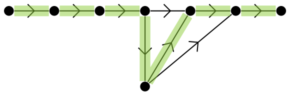

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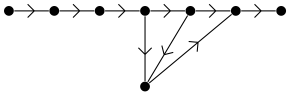

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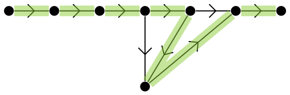

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- Take your favorite result in Ramsey theory, and prove (or disprove!) a directed version of it.

Thank you!