Bounds on Ramsey numbers

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December 9, 2024

Outline

What is Ramsey theory?

Fermat's last theorem and Ramsey numbers

Upper bounds on Ramsey numbers

Lower bounds on Ramsey numbers

Ramsey theory and random graphs

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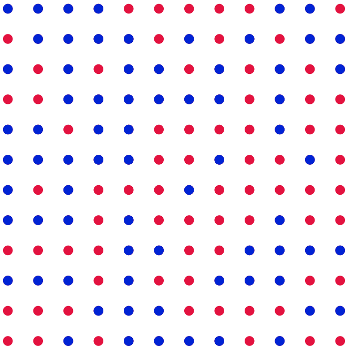
Given an $N \times N$ grid, if the points are colored red or blue, how large of a monochromatic subgrid can we find?



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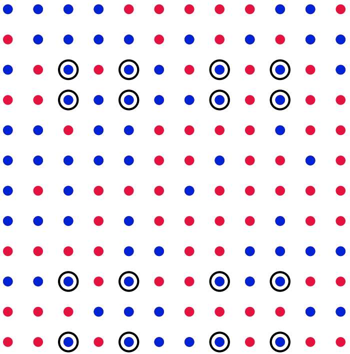

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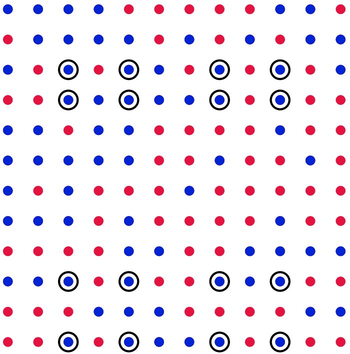

2-coloring = partition into two parts

monochromatic = entirely within one part

We are interested in the worst case, adversarial colorings.

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Theorem (Kővári–Sós–Turán 1954, Chvátal 1969)

In any 2-coloring of an $N \times N$ *grid, there is a* $\log N \times \log N$ *monochromatic subgrid*.

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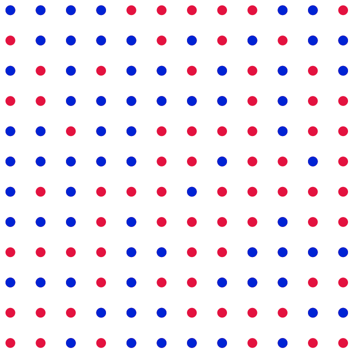

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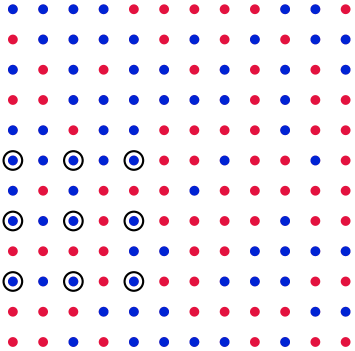

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Theorem (Gallai 1945, Witt 1952, Shelah 1988)

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Quantitative question: How quickly does f grow?

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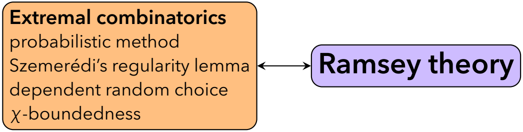

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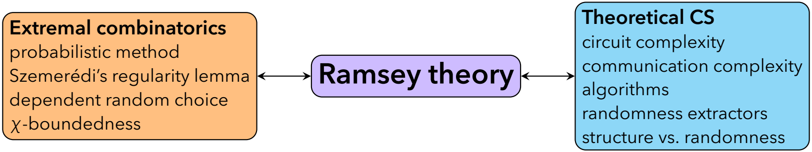

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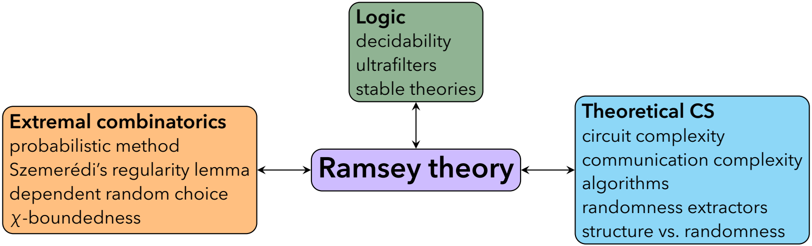

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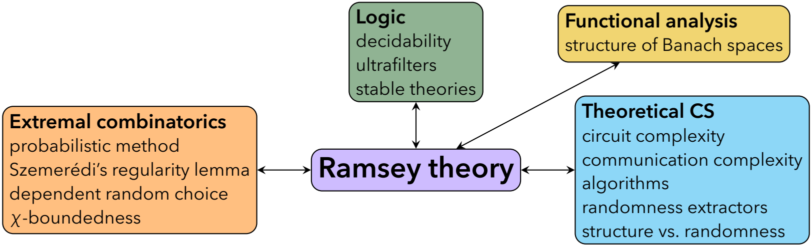

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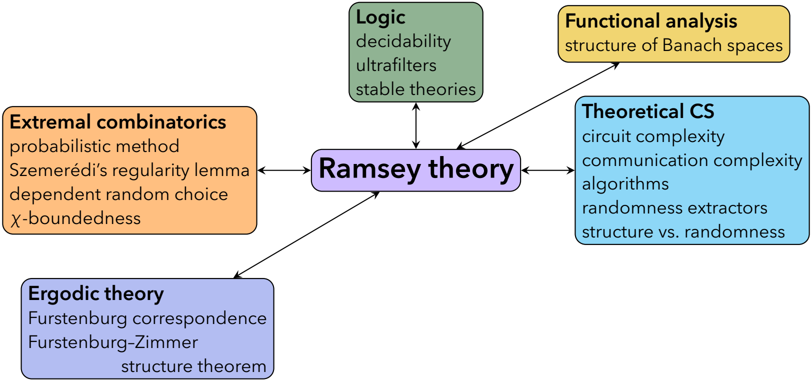

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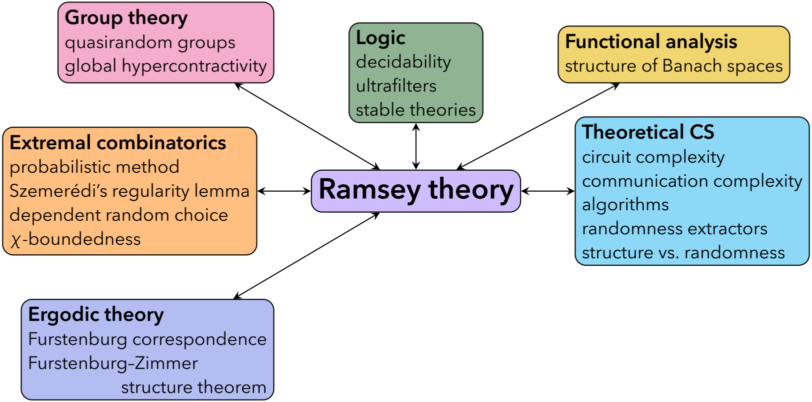

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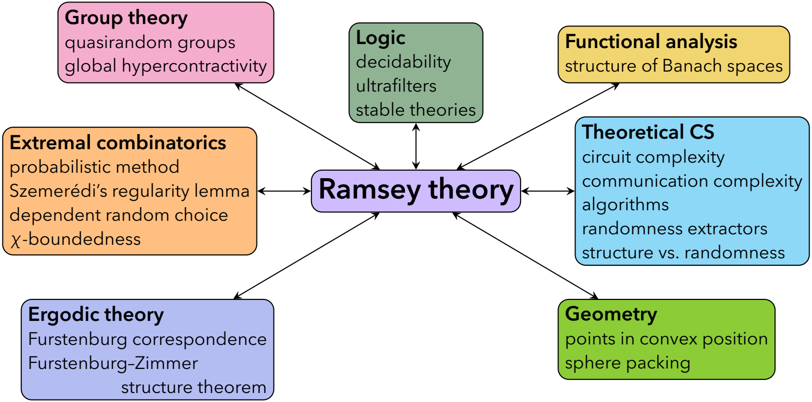

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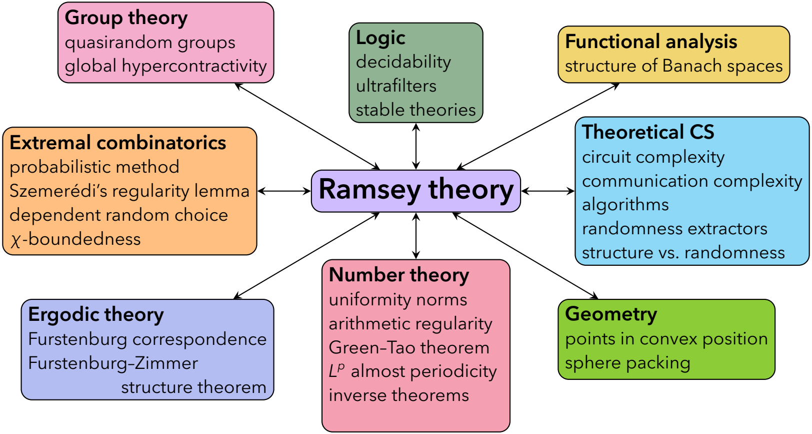

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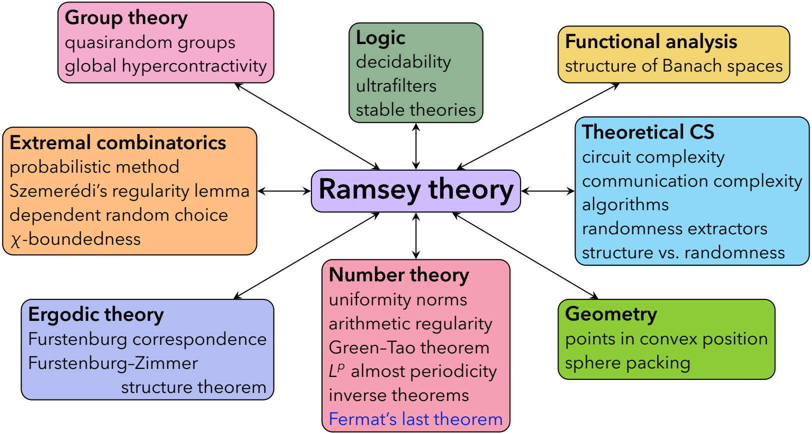

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Schur: "I will show how Dickson's theorem follows almost immediately from a very simple lemma, which belongs more to combinatorics than to number theory."

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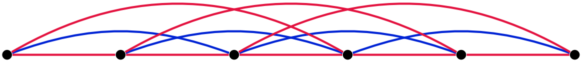

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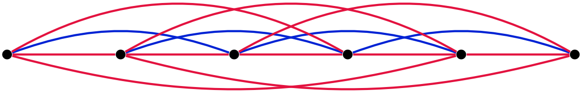

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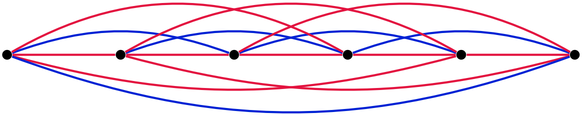

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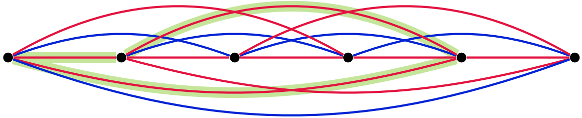

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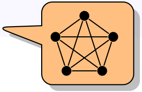

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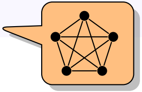

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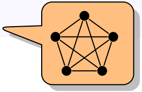

How large?

The Ramsey number $R(t;q)$ is the least N for which Ramsey's theorem holds.

Classical bounds on Ramsey numbers

Ramsey (1929): Fix integers t and q. If N is sufficiently large, then in any q-coloring of $E(K_N)$, there is a monochromatic K_t.

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Theorem (Erdős 1947)

$$R(t;2) > 2^{t/2}.$$

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For a lower bound on $R(t;q)$ we need a construction: a coloring of $E(K_N)$ with no monochromatic K_t. These are hard to find!

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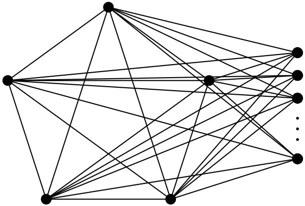

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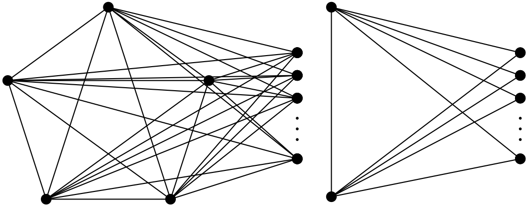

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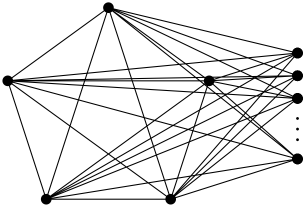

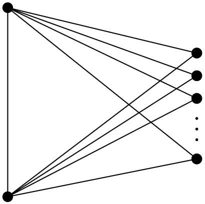

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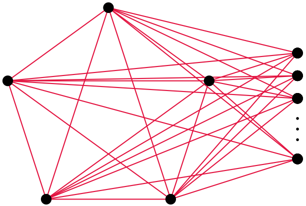

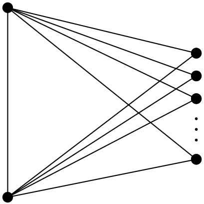

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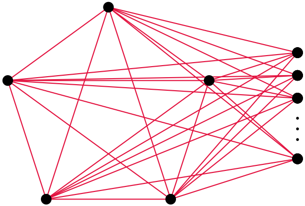

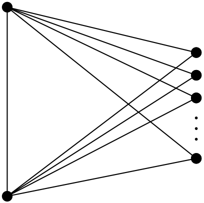

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In the m "page" vertices, it suffices to find a blue K_t or a red K_{t-s}.

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This result is still too weak to improve the bound $R(t;2) < 4^t$.

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The best known upper bound is essentially $(q^q)^t$ – a very big gap!

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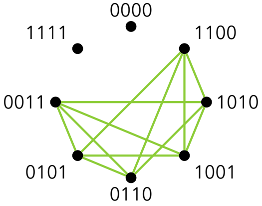

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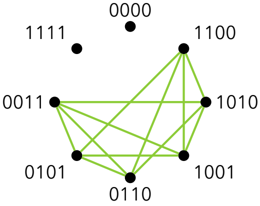

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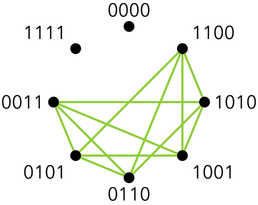

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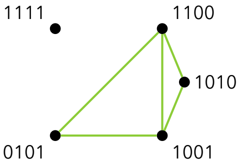

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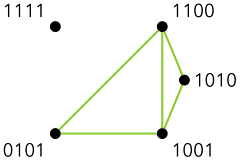

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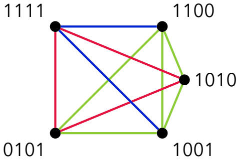

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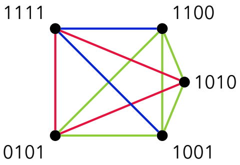

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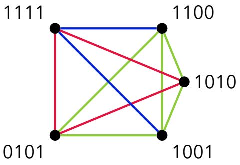

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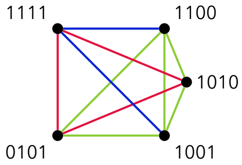

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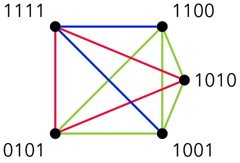

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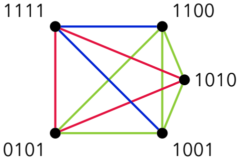

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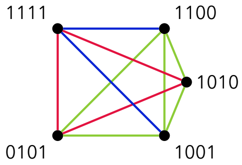

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This works over larger fields, but the bounds aren't very good.

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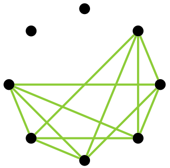

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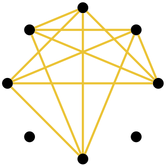

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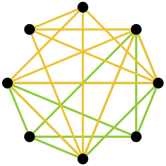

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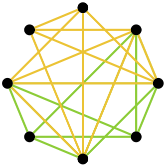

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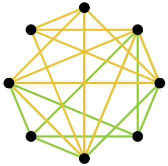

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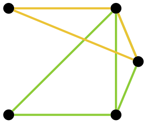

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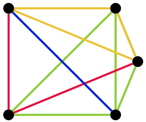

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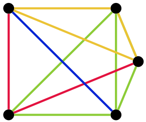

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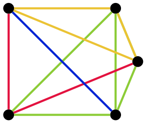

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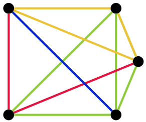

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How are we picking $p > 1$???

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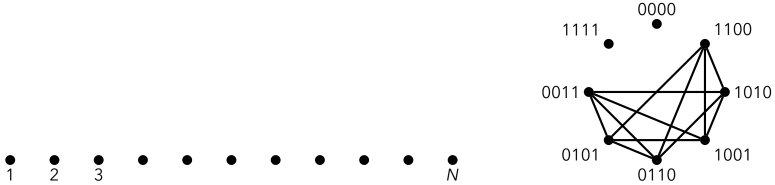

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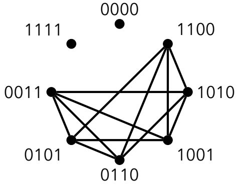

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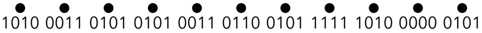

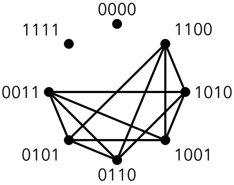

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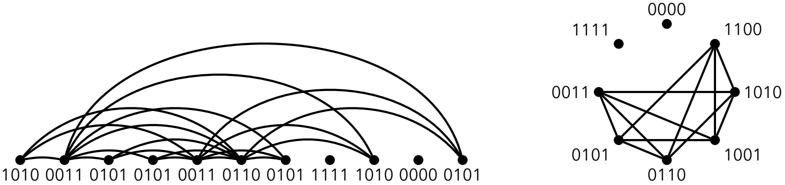

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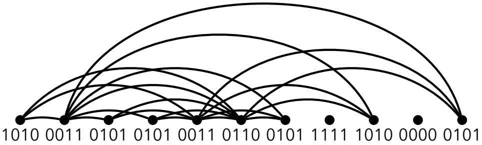

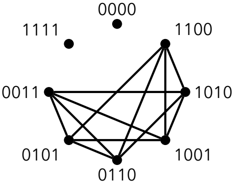

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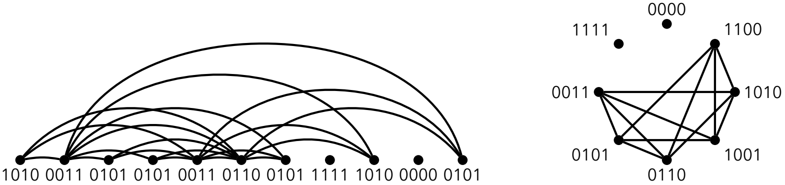

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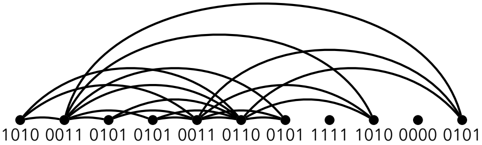

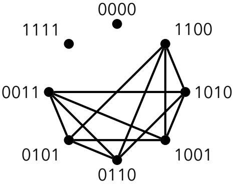

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So the above argument works for any p, if interpreted correctly.

Putting it all together

Theorem (W. 2021)

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By optimizing this technique, Sawin proved that

$$R(t;q) > \left(2^{0.383q-0.267}\right)^t.$$

Outline

What is Ramsey theory?

Fermat's last theorem and Ramsey numbers

Upper bounds on Ramsey numbers

Lower bounds on Ramsey numbers

Ramsey theory and random graphs

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An exact answer was conjectured by Kohayakawa–Kreuter (1997).

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Assuming the conjecture, one obtains a three-line proof of Folkman's theorem with good bounds.

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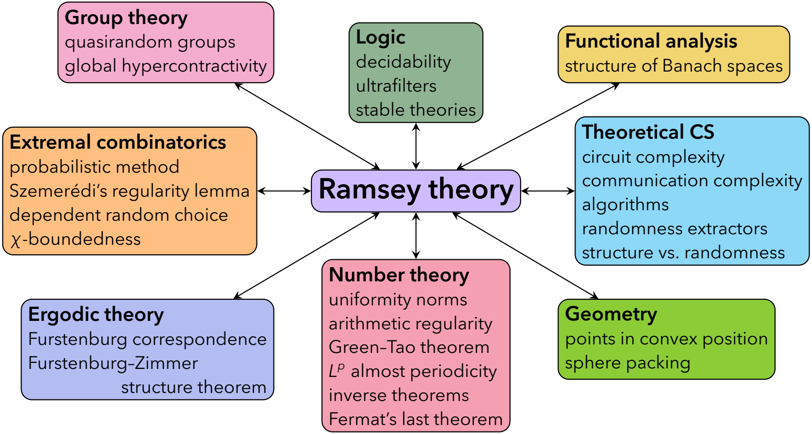
The proof uses max-flow min-cut, ideas from matroid theory,…

Thank you!


Ramsey theory
Group theory
quasirandom groups
global hypercontractivity
Logic
decidability
ultrafilters
stable theories
Functional analysis
structure of Banach spaces
Extremal combinatorics
probabilistic method
Szemerédi's regularity lemma
dependent random choice
χ-boundedness
Theoretical CS
circuit complexity
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algorithms
randomness extractors
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Ergodic theory
Furstenburg correspondence
Furstenburg–Zimmer structure theorem
Number theory
uniformity norms
arithmetic regularity
Green–Tao theorem
L^p almost periodicity
inverse theorems
Fermat's last theorem
Geometry
points in convex position
sphere packing