New perspectives on the uncertainty principle

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Uncertainty: that is appropriate for the matters of this world.

Joel and Ethan Coen
The Ballad of Buster Scruggs

Background

Heisenberg (1927): It is impossible to simultaneously determine a particle's position and momentum to arbitrary precision.

Kennard, Weyl (1927–8): Mathematical formalism:

$$\sigma_x \sigma_p \geq \frac{h}{4\pi}.$$

Also applies to any pair of canonically conjugate variables: quantities related by a Fourier transform.

From physics to math: relations to functional analysis, PDEs, microlocal analysis, wavelets, signal processing,...

The Fourier transform

Continuous

Given $f : \mathbb{R} \to \mathbb{C}$, we define

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, \mathrm{d}x.$$

We can recover f from $\hat{f}$ as

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} \, \mathrm{d}\xi.$$

Discrete

For $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$,

$$\hat{f}(\xi) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} f(x) e^{-2\pi i x \xi / N}$$

and we can recover

$$f(x) = \frac{1}{\sqrt{N}} \sum_{\xi=0}^{N-1} \hat{f}(\xi) e^{2\pi i x \xi / N}.$$

Throughout, f is "nice enough".

It suffices that $f, \hat{f} \in L^1$.

The uncertainty principle(s)

For $f : \mathbb{R} \to \mathbb{C}$

- If f has compact (finite-measure) support, then $\hat{f}$ does not.
- Hardy: f and $\hat{f}$ cannot both decay faster than e^{-x^2}.
- Heisenberg:

$$V(f)V(\hat{f}) \geq \frac{\|f\|_2^2 \|\hat{f}\|_2^2}{16\pi^2},$$

where $V(f) = \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx$.

- Hirschman, Beckner:

$$H(f) + H(\hat{f}) \geq \log \frac{e}{2},$$

where $\|f\|_2 = 1$ and $H(f) = \int_{\mathbb{R}} |f(x)|^2 \log |f(x)|^2 \, dx$.

For $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$

- Donoho–Stark:

$$|\mathrm{supp}(f)||\mathrm{supp}(\hat{f})| \geq N.$$

- If $\|f|_S\|_2 \geq (1-\varepsilon)\|f\|_2$ and $\|\hat{f}|_T\|_2 \geq (1-\delta)\|\hat{f}\|_2$, then

$$|S||T| \geq N(1-\varepsilon-\delta)^2.$$

- Biró, Meshulam, Tao: If N is prime,

$$|\mathrm{supp}(f)| + |\mathrm{supp}(\hat{f})| \geq N+1.$$

- Dembo–Cover–Thomas:

$$H(f) + H(\hat{f}) \geq \log N,$$

where $\|f\|_2 = 1$ and $H(f) = \sum_{x=0}^{N-1} |f(x)|^2 \log |f(x)|^2$.

Outline of the talk

Introduction

Our new perspective on the uncertainty principle

Examples

Interlude: generality and extensions

More examples

Conclusion

The primary uncertainty principle

We can bound $\|\hat{f}\|_\infty$ by $\|f\|_1$: observe that for any $\xi \in \mathbb{R}$,

$$|\hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, \mathrm{d}x \right| \leq \int_{-\infty}^{\infty} |f(x)| |e^{-2\pi i x \xi}| \, \mathrm{d}x = \|f\|_1$$

which implies that

$$\|\hat{f}\|_\infty = \sup_{\xi \in \mathbb{R}} |\hat{f}(\xi)| \leq \|f\|_1.$$

Similarly, $\|f\|_\infty \leq \|\hat{f}\|_1$. Multiplying these together, we find

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \geq 1. \tag{$\star$}$$

$\|f\|_1 / \|f\|_\infty$ measures localization; this is the uncertainty principle!

How to derive other uncertainty principles

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \geq 1 \qquad (\star)$$

Say we wish to prove an uncertainty principle

$$L(f) \cdot L(\hat{f}) \geq C$$

for some "measure of localization" L.

$(\star)$ contains all the "uncertainty". It suffices to prove

$$L(g) \geq c\frac{\|g\|_1}{\|g\|_\infty} \qquad \text{or} \qquad L(g) \geq \left(\frac{\|g\|_1}{\|g\|_\infty}\right)^c \qquad \text{or} \qquad \ldots$$

for any single function g. Then plug in $g = f, g = \hat{f}$.

Example 1: Support size

For any $g : \mathbb{R} \to \mathbb{C}$,

$$\begin{aligned} \|g\|_1 &= \int_{\mathbb{R}} |g(x)|\,\mathrm{d}x \\ &= \int_{\mathrm{supp}(g)} |g(x)|\,\mathrm{d}x \\ &\leq |\mathrm{supp}(g)| \|g\|_\infty \end{aligned}$$

where $|\cdot|$ is Lebesgue measure. So

$$|\mathrm{supp}(g)| \geq \frac{\|g\|_1}{\|g\|_\infty}.$$

Therefore

$$\begin{aligned} |\mathrm{supp}(f)||\mathrm{supp}(\hat{f})| &\geq \frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \\ &\geq 1. \end{aligned}$$

If $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, then

$$\|\hat{f}\|_\infty \leq \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |f(x)| = \frac{\|f\|_1}{\sqrt{N}}.$$

So in the discrete setting,

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \geq N.$$

Therefore, the same argument gives the Donoho–Stark UP:

For non-zero $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$,

$$|\mathrm{supp}(f)| \cdot |\mathrm{supp}(\hat{f})| \geq N.$$

Example 2: Approximate support

Say that g is *ε-supported* on $E \subset \mathbb{R}$ if

$$\int_E |g(x)|\,\mathrm{d}x \geq (1-\varepsilon)\|g\|_1.$$

Theorem (Williams)

If $f : \mathbb{R} \to \mathbb{C}$ is ε-supported on S and $\hat{f}$ is δ-supported on T, then

$$|S||T| \geq (1-\varepsilon)(1-\delta).$$

Proof.

If g is ε-supported on E, then

$$(1-\varepsilon)\|g\|_1 \leq \int_E |g(x)|\,\mathrm{d}x \leq |E|\|g\|_\infty \quad \Longrightarrow \quad |E| \geq (1-\varepsilon)\frac{\|g\|_1}{\|g\|_\infty}.$$

So

$$|S||T| \geq \left((1-\varepsilon)\frac{\|f\|_1}{\|f\|_\infty}\right)\left((1-\delta)\frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty}\right) \geq (1-\varepsilon)(1-\delta). \qquad \square$$

How general is this?

All our results follow from the primary uncertainty principle

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \geq 1, \qquad (\star)$$

as well as "universal" bounds that hold for a single function.

($\star$) only needed that $\|\hat{f}\|_\infty \leq \|f\|_1$, $\|f\|_\infty \leq \|\hat{f}\|_1$.

Theorem

Let A, B be a linear operators with

$$\|A\|_{1\to\infty} \leq 1, \|B\|_{1\to\infty} \leq 1, \text{ and } \|BAf\|_\infty \geq k\|f\|_\infty.$$

Then

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|Af\|_1}{\|Af\|_\infty} \geq k.$$

We call such operators *k-Hadamard*. Examples from coding theory, block designs, quantum algebra, fractional Fourier transforms…

Example 3: Other norms

Theorem

For any $1 \leq p \leq \infty$ *and any non-zero* $f : \mathbb{R} \to \mathbb{C}$,

$$\frac{\|f\|_1}{\|f\|_p} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_p} \geq 1.$$

Proof.

For any $g : \mathbb{R} \to \mathbb{C}$,

$$\|g\|_p^p = \int_{\mathbb{R}} |g(x)|^p \, \mathrm{d}x \leq \|g\|_\infty^{p-1} \int_{\mathbb{R}} |g(x)| \, \mathrm{d}x = \|g\|_\infty^{p-1} \|g\|_1.$$

So $\|g\|_1^{p-1} \|g\|_p^p \leq \|g\|_\infty^{p-1} \|g\|_1^p$. Therefore,

$$\frac{\|g\|_1}{\|g\|_p} \geq \left(\frac{\|g\|_1}{\|g\|_\infty} \right)^{\frac{p-1}{p}} \implies \frac{\|f\|_1}{\|f\|_p} \frac{\|\hat{f}\|_1}{\|\hat{f}\|_p} \geq \left(\frac{\|f\|_1}{\|f\|_\infty} \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \right)^{\frac{p-1}{p}} \geq 1.$$

□

Example 4: The Heisenberg uncertainty principle

$$V(f)V(\hat{f}) \geq C\|f\|_2^2\|\hat{f}\|_2^2, \text{ where } V(f) = \int_{\mathbb{R}} x^2|f(x)|^2\,\mathrm{d}x.$$

Here, we will use $\frac{\|f\|_1}{\|f\|_2}\frac{\|\hat{f}\|_1}{\|\hat{f}\|_2} \geq 1$ as a primary uncertainty principle.

So we want a lower bound on $V(g)$ in terms of $\|g\|_1/\|g\|_2$.

Lemma: $\frac{V(g)}{\|g\|_2^2} \geq c\left(\frac{\|g\|_1}{\|g\|_2}\right)^4$.

Proof sketch: Let $T = \frac{1}{8}(\|g\|_1/\|g\|_2)^2$, so that $\int_{-T}^{T}|g| \leq \frac{1}{2}\|g\|_1$. Then

$$\frac{1}{2}\|g\|_1 \leq \int_{|x|>T}\frac{x}{x}|g(x)|\,\mathrm{d}x \leq \underbrace{\left(\int_{|x|>T}\frac{1}{x^2}\,\mathrm{d}x\right)^{\frac{1}{2}}}_{\sqrt{2/T}}\underbrace{\left(\int_{\mathbb{R}}x^2|g(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}}_{\sqrt{V(g)}}.$$

Using the same ideas, we can prove uncertainty relations for other moments and norms of $|f|, |\hat{f}|$; similar to results of Cowling–Price.

The constant we get is not optimal. This is probably an unavoidable shortcoming of this technique.

Summary

- Many (but not all!) uncertainty principles follow from a simple and general framework.

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty} \geq 1$$

- To prove other uncertainty principles, simply prove a bound that holds for any single function.
- Standard proofs of many of these results use special properties of the Fourier transform; we only use basic calculus facts.
- Shortcomings: rarely gets the correct constant, cannot prove all uncertainty principles. But these lead to interesting questions…
- The only property of the Fourier transform we used is that it and its inverse are bounded $L^1 \to L^\infty$. Thus, this works in much greater generality.

Other operators for which this works

The only property of the Fourier transform we used is that it and its inverse are bounded $L^1 \to L^\infty$. Thus, this works in much greater generality.

- Continuous case:
  - Linear canonical transforms: given $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $b \neq 0$,

$$(L_M f)(\xi) = \frac{1}{\sqrt{ib}} \int_{-\infty}^{\infty} f(x) e^{i\pi(d\xi^2 - 2x\xi + ax^2)/b} \, \mathrm{d}x.$$

- Discrete case:
  - We can obtain uncertainty principles for many "structured" linear operators: Hadamard matrices, conference matrices, incidence matrices from discrete geometry, error-correcting codes...
  - We can also get uncertainty principles for random matrices. The Fourier transform isn't so special—almost all matrices satisfy uncertainty!

Further extensions and open problems

Theorem

Let A, B be linear operators with

$$\|A\|_{1\to\infty} \le 1, \|B\|_{1\to\infty} \le 1, \text{ and } \|BAf\|_\infty \ge k\|f\|_\infty.$$

Then

$$\frac{\|f\|_1}{\|f\|_\infty} \cdot \frac{\|Af\|_1}{\|Af\|_\infty} \ge k.$$

- Arbitrary pairs of norms – very useful for non-abelian groups.
- Multiple operators $A_1, \ldots, A_n$ – are there any applications?
- Do some non-linear operators have uncertainty principles?
- Can one prove the multidimensional Heisenberg uncertainty principle with these techniques? If $f : \mathbb{R}^n \to \mathbb{C}$, then

$$\left(\int_{\mathbb{R}^n} \|x\|_2^2 |f(x)|^2 \, \mathrm{d}x\right) \left(\int_{\mathbb{R}^n} \|\xi\|_2^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right) \ge \frac{n^2}{16\pi^2} \|f\|_2^2 \|\hat{f}\|_2^2.$$

  The main interest is getting the correct dependence on n.

Thank you!