New perspectives on the uncertainty principle

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Joint with Avi Wigderson

March 26, 2021

Uncertainty: that is appropriate for the matters of this world.

Joel and Ethan Coen
The Ballad of Buster Scruggs

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From physics to math: relations to functional analysis, PDEs, microlocal analysis, wavelets, signal processing,...

The Fourier transform

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and we can recover

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Throughout, f is "nice enough".
It suffices that $f, \hat{f} \in L^1$.

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- If f has compact support, then $\hat{f}$ does not.

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$$V(f)V(\hat{f}) \geq \frac{\|f\|_2^2 \|\hat{f}\|_2^2}{16\pi^2},$$

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- Hirschman, Beckner:

$$H(f) + H(\hat{f}) \geq \log \frac{e}{2},$$

  where $\|f\|_2 = 1$ and
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- Dembo–Cover–Thomas:

$$H(f) + H(\hat{f}) \geq \log N,$$

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Outline of the talk

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Therefore, the same argument gives the Donoho–Stark UP:

For non-zero $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$,

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So

$$|S||T| \geq \left((1-\varepsilon)\frac{\|f\|_1}{\|f\|_\infty}\right)\left((1-\delta)\frac{\|\hat{f}\|_1}{\|\hat{f}\|_\infty}\right) \geq (1-\varepsilon)(1-\delta). \qquad \square$$

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We call such operators *k-Hadamard*. Examples from coding theory, block designs, quantum algebra, fractional Fourier transforms...

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Example 4: The Heisenberg uncertainty principle

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For non-zero $f : \mathbb{R} \to \mathbb{C}$,

$$\left(\int_{\infty}^{\infty} x^2 |f(x)|^2 \, \mathrm{d}x\right) \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right) \geq C \|f\|_2^2 \|\hat{f}\|_2^2.$$

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Using the same ideas, we can prove uncertainty relations for other moments and norms of $|f|, |\hat{f}|$; similar to results of Cowling–Price.

The constant we get is not optimal. This is probably an unavoidable shortcoming of this technique.

Summary

- Many (but not all!) uncertainty principles follow from a simple and general framework.

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- The only property of the Fourier transform we used is that it and its inverse are bounded $L^1 \to L^\infty$. Thus, this works in much greater generality.

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  - Linear canonical transforms: given $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $b \neq 0$,

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  - We can obtain uncertainty principles for many "structured" linear operators: Hadamard matrices, conference matrices, incidence matrices from discrete geometry, error-correcting codes…
  - We can also get uncertainty principles for random matrices. The Fourier transform isn't so special–almost all matrices satisfy uncertainty!

Further extensions and open problems

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- Multiple operators $A_1, \ldots, A_n$ – are there any applications?
- Do some non-linear operators have uncertainty principles?
- Can one prove the multidimensional Heisenberg uncertainty principle with these techniques? If $f : \mathbb{R}^n \to \mathbb{C}$, then

$$\left(\int_{\mathbb{R}^n} \|x\|_2^2 |f(x)|^2 \, \mathrm{d}x\right) \left(\int_{\mathbb{R}^n} \|\xi\|_2^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi\right) \geq \frac{n^2}{16\pi^2} \|f\|_2^2 \|\hat{f}\|_2^2.$$

The main interest is getting the correct dependence on n.

Thank you!