# Ramsey numbers upon vertex deletion

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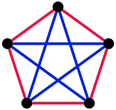

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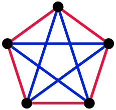

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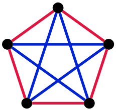

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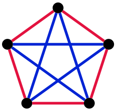

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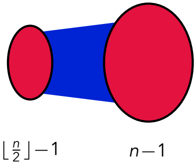

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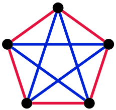

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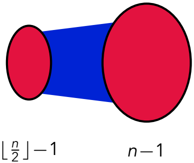

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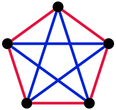

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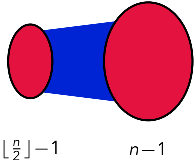


**Question:** Say a coloring is $\varepsilon$-balanced if both colors have $\geq \varepsilon\binom{N}{2}$ edges. Is every Ramsey coloring $\varepsilon$-balanced, where $\varepsilon > 0$ is fixed?

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  If $G$ has an $\varepsilon$-balanced Ramsey coloring, then $r(G) \leq C(\varepsilon) \cdot r(H)$.

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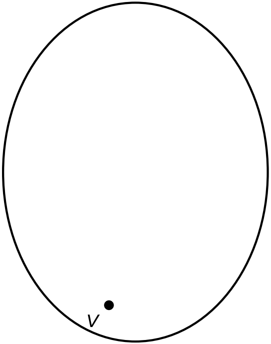

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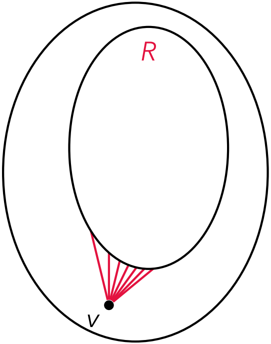

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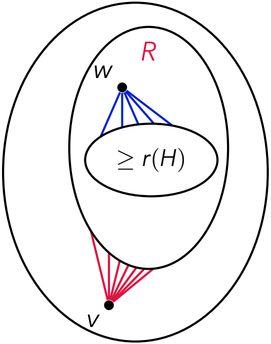

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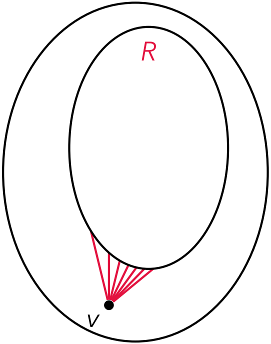

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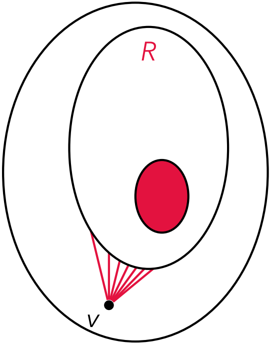

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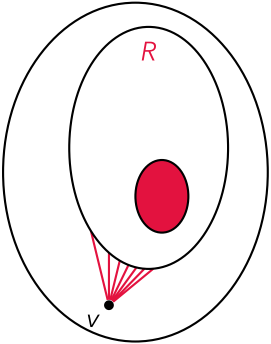


## Theorem (W. 2022)

*Under the same hypotheses,* $r(G) \leq C\sqrt{n \log n} \cdot r(H)$.

# Balanced colorings

**Recall:** A Ramsey coloring for $G$ has no monochromatic copy of $G$ and is on $r(G) - 1$ vertices.
An $\varepsilon$-balanced coloring of $K_N$ has $\geq \varepsilon\binom{N}{2}$ edges in both colors.

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## Proposition

*If $G$ has an $\varepsilon$-balanced Ramsey coloring, then $r(G) \leq \frac{100}{\varepsilon} \cdot r(H)$.*

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*If $G$ has $\ge \delta n^2$ edges, then $r(G) \le C(\delta) \cdot r(H)$.*

This follows immediately from the Proposition and from:

**Theorem (Erdős–Szemerédi 1972):** If $G$ has $\ge \delta n^2$ edges, then every Ramsey coloring for $G$ is $\varepsilon(\delta)$-balanced.

# Motivations

## Conjecture (Conlon–Fox–Sudakov 2020)

*Delete a single vertex of $G$ to obtain $H$. Then $r(G) \leq C \cdot r(H)$ for some absolute constant $C > 0$.*

Why should we believe this conjecture?

- **It's true "on average".**
  Build $G$ up one vertex at a time, as $G_1, G_2, \ldots, G_n = G$.
  $r(G_1) = 1$ and $r(G_n) \leq 4^n$, so $r(G_{i+1}) \leq 4 \cdot r(G_i)$ for an average $i$.
- **It's "almost" true.**
  $r(G) \leq 2n \cdot r(H)$.
- **It's true for dense graphs.**
  If $G$ has $\geq \delta n^2$ edges, then $r(G) \leq C(\delta) \cdot r(H)$.
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  If $G$ has $o(n^2)$ edges, then $r(G) \leq 2^{o(n)}$.
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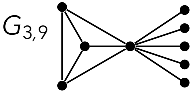

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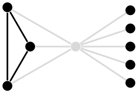

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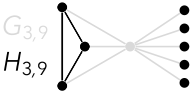


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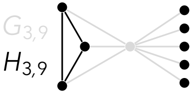


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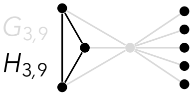

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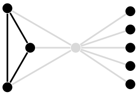


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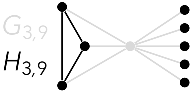


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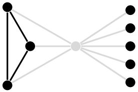


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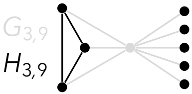


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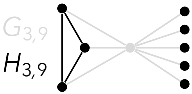


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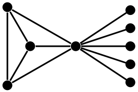


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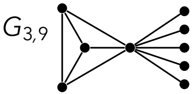


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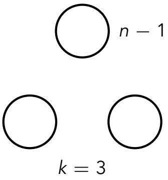

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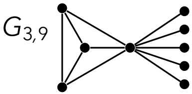


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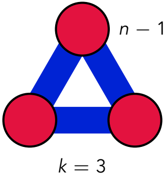

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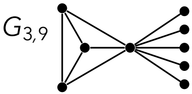


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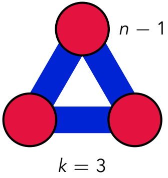

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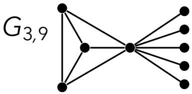


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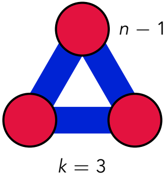

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In general, the Turán coloring is one of very few general-purpose constructions in Ramsey theory. Finding new constructions could lead to progress on many questions.

# Philosophy

The heart of the proof is connectivity. The Turán coloring gives a good lower bound on $r(G_{k,n})$ because $G_{k,n}$ is connected.

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Plausibly, the “right” notion here is expansion. However, I don’t know an analogue of the Turán coloring that “detects” expansion.

# Open problems I

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## Theorem (W. 2022)

*Delete a vertex of G to obtain H. Then* $r(G)/r(H) = O(\sqrt{n \log n})$.
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The upper bound is proved in the same way, but it's uninteresting because $r(G;q)$ is already typically exponential in $n$.

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*Delete an edge of $G$ to obtain $H$. Then $r(G) \leq C \cdot r(H)$.*

**Intuition:** Deleting an edge can't split $G$ into many components.

# Thank you!