Ramsey numbers upon vertex deletion

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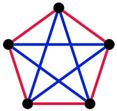

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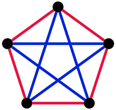

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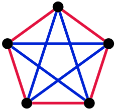

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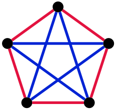

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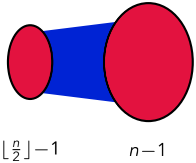

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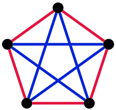

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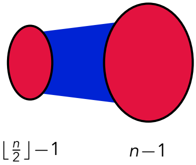
A Ramsey coloring for the path $G = P_n$. Here $N \approx \frac{3}{2}n$, and there are $\approx \frac{5}{9}\binom{N}{2}$ red and $\approx \frac{4}{9}\binom{N}{2}$ blue edges.



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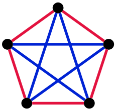

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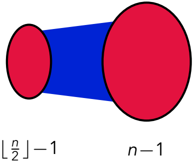

Question: Say a coloring is ε-balanced if both colors have $\geq \varepsilon\binom{N}{2}$ edges. Is every Ramsey coloring ε-balanced, where $\varepsilon > 0$ is fixed?

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  For example, we expect $r(K_n) - r(K_{n-1}) > 2^{cn}$, but the state of the art is $r(K_n) - r(K_{n-1}) \geq 4n - 8$. [Burr–Erdős–Faudree-Schelp]

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- **It's related to the warmup question.**
  If G has an ε-balanced Ramsey coloring, then $r(G) \leq C(\varepsilon) \cdot r(H)$.

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Fix a coloring on $2n \cdot r(H)$ vertices and a vertex v. We seek a monochromatic G.

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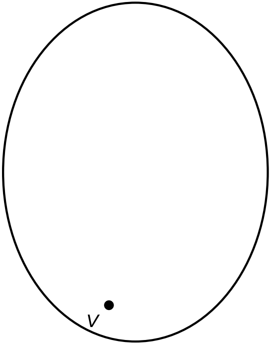

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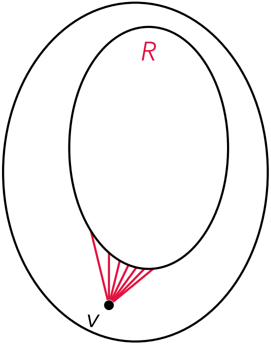

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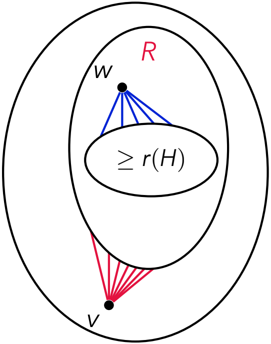

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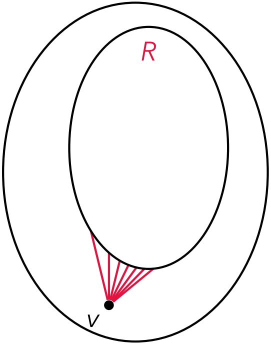

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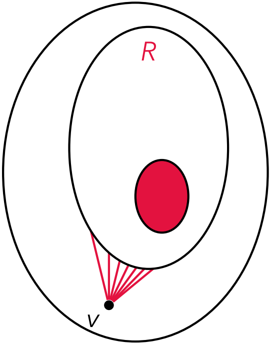

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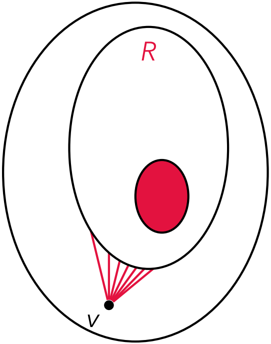

Theorem (W. 2022)

Under the same hypotheses, $r(G) \leq C\sqrt{n \log n} \cdot r(H)$.

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Recall: A Ramsey coloring for G has no monochromatic copy of G and is on $r(G) - 1$ vertices.
An ε-balanced coloring of K_N has $\geq \varepsilon\binom{N}{2}$ edges in both colors.

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If G has an ε-balanced Ramsey coloring, then $r(G) \leq \frac{16}{\varepsilon^2} \cdot r(H)$.

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If G has $\geq \delta n^2$ edges, then $r(G) \leq C(\delta) \cdot r(H)$.

This follows immediately from the Proposition and from:

Theorem (Erdős–Szemerédi 1972): If G has $\geq \delta n^2$ edges, then every Ramsey coloring for G is $\varepsilon(\delta)$-balanced.

Motivations

Conjecture (Conlon–Fox–Sudakov 2020)

Delete a single vertex of G to obtain H. Then $r(G) \leq C \cdot r(H)$ for some absolute constant $C > 0$.

Why should we believe this conjecture?

- **It's true "on average".**
  Build G up one vertex at a time, as $G_1, G_2, \ldots, G_n = G$.
  $r(G_1) = 1$ and $r(G_n) \leq 4^n$, so $r(G_{i+1}) \leq 4 \cdot r(G_i)$ for an average i.
- **It's "almost" true.**
  $r(G) \leq 2n \cdot r(H)$.
- **It's true for dense graphs.**
  If G has $\geq \delta n^2$ edges, then $r(G) \leq C(\delta) \cdot r(H)$.
- **It "should be even truer" for sparse graphs.**
  If G has $o(n^2)$ edges, then $r(G) \leq 2^{o(n)}$.
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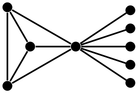

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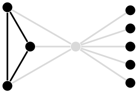

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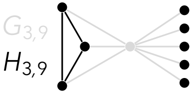

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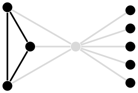

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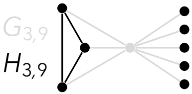

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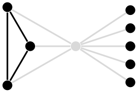

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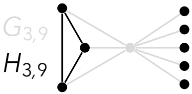

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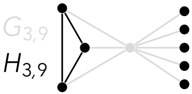

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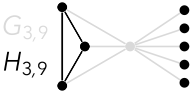

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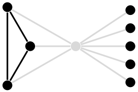

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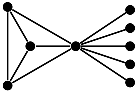

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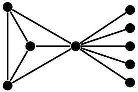

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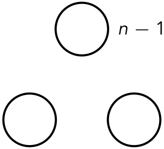

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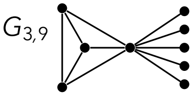

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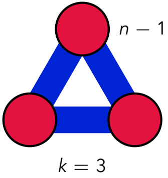

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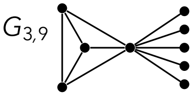

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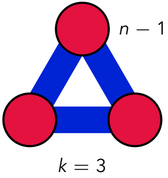

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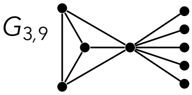

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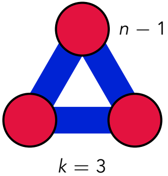

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Plausibly, the "right" notion here is expansion. However, I don't know an analogue of the Turán coloring that "detects" expansion.

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Theorem (W. 2022)

For $q \geq 3$, *we have* $r(G; q)/r(H; q) < 2^{O(n)}$. *There exists G with* $r(G; q)/r(H; q) > n^{\theta}$ *for some* $\theta = \theta(q) > 0$.

The upper bound is proved in the same way, but it's uninteresting because $r(G;q)$ is already typically exponential in n.

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Intuition: Deleting an edge can't split G into many components.

Thank you!