Exercises (recommended)

1. Prove, as claimed in class, that $\frac{2^n-1}{n} \leqslant \text{magic}(n) < 2^{n-1}$ for all integers n.
2. What does Turán's theorem mean in case $r = 2$? Is the theorem true in that case?
3. Find a general formula for $t_{r-1}(n)$, in terms of n, r, and $s \coloneqq n \pmod{r-1}$.
4. Prove that $T_{r-1}(n)$ maximizes number of edges among all complete $(r-1)$-partite graphs (that is, that any complete $(r-1)$-partite graph with parts of sizes *different* from $\lfloor n/(r-1) \rfloor$ or $\lceil n/(r-1) \rceil$ has fewer edges than $T_{r-1}(n)$).
5. Provide an alternative proof of Turán's theorem by induction on n (with inductive steps of size 1) by deleting a vertex of minimum degree.
6. Let G be an n-vertex graph. Recall that the *independence number* of G, denoted $\alpha(G)$, is the size of the largest set of vertices in G containing no edge. Let Δ be the maximum degree of G, and let d be the average degree of G.
   (a) Prove that $\chi(G) \leqslant \Delta(G) + 1$. Conclude that $\alpha(G) \geqslant n/(\Delta + 1)$.
   (b) Using Turán's theorem, prove that $\alpha(G) \geqslant n/(d+1)$. Note that this is a (much!) stronger result.

Problems (optional)

1. In this problem, you will show that $\text{magic}(n) = \Omega(2^n/\sqrt{n})$, following the argument of Erdős and Moser. This problem assumes some familiarity with probability, specifically Chebyshev's inequality.
   (a) Let $a_1, \ldots, a_n \in [\![M]\!]$, and suppose that all subsets have distinct sums. Let $\xi_1, \ldots, \xi_n$ be independent random variables, each taking on the values 0 or 1 with probability $\frac{1}{2}$, and let
   $$X = \sum_{i=1}^{n} \xi_i a_i.$$
   Prove that for every integer x, we have that $\Pr(X = x) \leqslant 2^{-n}$.
   (b) Prove that $\text{Var}(X) \leqslant M^2 n/4$.
   (c) Let $\lambda > 1$ be some parameter. Using Chebyshev's inequality, plus the previous two parts, prove that
   $$1 - \frac{1}{\lambda^2} \leqslant \Pr\left(|X - \mathbb{E}[X]| < \frac{\lambda M\sqrt{n}}{2}\right) \leqslant 2^{-n}\left(\lambda M\sqrt{n} + 1\right).$$

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(d) By picking λ appropriately, prove that $M = \Omega(2^n/\sqrt{n})$. Deduce that magic$(n) = \Omega(2^n/\sqrt{n})$. What is the best constant factor you can obtain by optimizing λ?

✣ 2. (a) Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^d$ with $\|v_i\| \geqslant 1$ for all i, where $\|\cdot\|$ denotes the usual Euclidean length of a vector. Prove that there are at most $\lfloor \frac{n^2}{4} \rfloor$ pairs v_i, v_j with $\|v_i + v_j\| < 1$.

⋆ (b) Fix a probability distribution on $\mathbb{R}^d$, and let X, Y be two independent random vectors drawn according to this distribution. Prove that

$$\Pr(\|X + Y\| \geqslant 1) \geqslant \frac{1}{2} \Pr(\|X\| \geqslant 1)^2.$$

(c) Find a probability distribution on $\mathbb{R}^d$ for which the above bound is tight.

3. A *directed graph* is a graph in which every edge is assigned one of the two possible directions. In a directed graph, we allow *anti-parallel edges*, i.e. $x \to y$ and $y \to x$ may both be edges in the same directed graph. An *oriented graph* is a directed graph without anti-parallel edges.

   (a) A *cyclic triangle* is the oriented graph on 3 vertices with edges $x \to y, y \to z, z \to x$. What is the maximum number of edges in an n-vertex oriented graph without a cyclic triangle?

   (b) A *transitive triangle* is the oriented graph on 3 vertices with edges $x \to y, y \to z, x \to z$. What is the maximum number of edges in an n-vertex oriented graph without a transitive triangle?

   (c) What is the maximum number of edges in an n-vertex *directed* graph without a cyclic triangle?

   (d) What is the maximum number of edges in an n-vertex *directed* graph without a transitive triangle?

Exercises (recommended)

1. Let $\mathcal{H}$ be a collection of graphs. We say that G is $\mathcal{H}$-*free* if G has no copy of any $H \in \mathcal{H}$, and we define

$$\mathrm{ex}(n, \mathcal{H}) = \max\{e(G) : G \text{ is an } n\text{-vertex } \mathcal{H}\text{-free graph}\}.$$

Assuming the Erdős–Stone–Simonovits theorem, prove that

$$\mathrm{ex}(n, \mathcal{H}) = \left(1 - \frac{1}{\chi(\mathcal{H}) - 1} + o(1)\right)\binom{n}{2},$$

where $\chi(\mathcal{H}) \coloneqq \min\{\chi(H) : H \in \mathcal{H}\}$.

2. By carefully analyzing the proof we saw in class, prove that $\mathrm{ex}(n, C_4) \leqslant \frac{n(\sqrt{4n-3}+1)}{4}$.

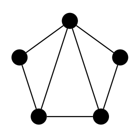
Hint: Start with the case that G is d-regular, where $d \geqslant \frac{\sqrt{4n-3}+1}{2}$.

3. Today we proved that for any graph H,

$$\mathrm{ex}(n, H) \geqslant t_{\chi(H)-1}(n), \qquad (*)$$

which in particular implies the lower bound in the Erdős–Stone–Simonovits theorem. In this problem, you'll see examples of graphs where inequality $(*)$ is not best possible, i.e. where the Turán graph $T_{\chi(H)-1}(n)$ has strictly fewer edges than $\mathrm{ex}(n, H)$.

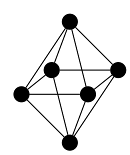
(a) Let H be the graph



Verify that $\chi(H) = 3$, so that inequality $(*)$ implies $\mathrm{ex}(n, H) \geqslant t_2(n) = \lfloor n^2/4 \rfloor$.

(b) Add some edges to the Turán graph $T_2(n)$ to prove that $\mathrm{ex}(n, H) \geqslant \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor$.

⋆ (c) Let O_3 be the graph corresponding to the octahedron, namely the graph



Verify that $\chi(O_3) = 3$. Add edges to $T_2(n)$ to prove that

$$\mathrm{ex}(n, O_3) \geqslant \left\lfloor \frac{n^2}{4} \right\rfloor + cn^{3/2},$$

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for some absolute constant $c > 0$.

Hint: You may assume the fact that I stated but didn't prove in class, namely that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$ (i.e. that we have a matching lower bound to the upper bound we proved).

(d) Why don't these examples violate the Erdős–Stone–Simonovits theorem?

4. Provide an alternative proof of Turán's theorem using a technique called *Zykov symmetrization*. Let G be a K_r-free n-vertex graph.

   (a) Pick two non-adjacent vertices $x, y \in V(G)$, and assume without loss of generality that $\deg(x) \geqslant \deg(y)$. Replace y with a *clone* of x, i.e. another vertex x' with the same neighborhood as x.

   (b) Repeat step (a) over and over until doing so no longer changes the graph (and prove that this must eventually happen).

   (c) Prove that the resulting graph when you get stuck is complete $(r-1)$-partite.

   (d) Conclude that $e(G) \leqslant t_{r-1}(n)$, with equality if and only if $G \cong T_{r-1}(n)$.

Problems (optional)

✣★1. Suppose $p_1, \ldots, p_n \in \mathbb{R}^2$ are n points in the plane. Prove that the number of *unit distances* among them (i.e. pairs $\{p_i, p_j\}$ with $\|p_i - p_j\| = 1$) is at most $O(n^{3/2})$.

Can you prove a stronger upper bound, or find a matching lower bound?

★2. Let G be an n-vertex triangle-free graph.

   (a) Suppose every vertex of G has degree greater than $2n/5$. Prove that G is bipartite.

   (b) Show that $2/5$ is the optimal constant in this theorem, that is, that for every n, there exists a non-bipartite triangle-free graph with minimum degree $\lfloor 2n/5 \rfloor$.

   ★★(c) Can you find generalizations of parts (a) and (b) for K_r-free graphs, $r > 3$?

✣3. In this problem you will prove Jensen's inequality in full generality.

   (a) A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

   Prove that if f is twice-differentiable and satisfies $f'' \geqslant 0$, then f is convex.

   (b) Suppose f is convex. Let $x_1, \ldots, x_n \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\lambda_1 + \cdots + \lambda_n = 1$. Prove that

$$\sum_{i=1}^n \lambda_i f(x_i) \geqslant f\left(\sum_{i=1}^n \lambda_i x_i\right)$$

   by induction on n. This is the general form of Jensen's inequality.

   (c) Prove that $f(x) = \binom{x}{r}$ is convex on the interval $[r, \infty)$ using part (a), and conclude the version of Jensen's inequality that I stated in class from part (b).

Exercises (recommended)

1. (a) Suppose that T is a tree with $t+1$ vertices, and G is a graph with minimum degree at least t. Prove that G contains a copy of T.
   Hint: Induction on t.

   (b) Let G be an n-vertex graph with m edges. Prove that there is a subgraph $G' \subseteq G$ with minimum degree strictly greater than m/n.
   Hint: Repeatedly delete vertices of degree $\leqslant m/n$.

   (c) Using parts (a) and (b), prove that if T is a tree with $t+1$ vertices, then
   $$\mathrm{ex}(n, T) < (t-1)n.$$

   (d) Prove that if n is divisible by t, then
   $$\mathrm{ex}(n, T) \geqslant \frac{(t-1)n}{2}.$$

   ? (e) Erdős and Sós conjectured that the lower bound in part (d) is best possible, i.e. that
   $$\mathrm{ex}(n, T) = \left\lfloor \frac{(t-1)n}{2} \right\rfloor$$
   for all $(t+1)$-vertex trees T. Can you prove or disprove this conjecture?

2. Let $K_{1,r}$ denote the star with r leaves. Determine $\mathrm{ex}(n, K_{1,r})$ for all n and r. Is your answer consistent with the Erdős–Sós conjecture from exercise 1? Is it consistent with the Kővári–Sós–Turán theorem we proved in class?

3. Recall that we defined
   $$m_2(H) = \max_{F \subseteq H} \frac{e(F)-1}{v(F)-2},$$
   and stated in class that $\mathrm{ex}(n, H) \geqslant \Omega(n^{2-1/m_2(H)})$ for all bipartite H.

   (a) Compute $m_2(C_{2\ell})$ for each $\ell \geqslant 2$. What lower bound on $\mathrm{ex}(n, C_{2\ell})$ do you get?

   (b) Compute $m_2(K_{s,t})$ for all $t \geqslant s \geqslant 2$. How does the resulting lower bound compare to the others we've discussed?

   (c) Compute $m_2(T)$ for any tree T. How does the resulting lower bound relate to exercise 1?

   ⋆ (d) Pick your favorite bipartite graph, and compute the lower and upper bounds coming from $m_2(H)$ and from finding H as a subgraph of $K_{s,t}$, respectively. Can you improve either of these bounds?

4. Using previous homework problems, prove the following fact. A graph H is a forest if and only if $\mathrm{ex}(n, H) \leqslant O(n)$.

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Problems (optional)

$\star$ 1. In this problem, you'll prove that $\mathrm{ex}(n,H) \geqslant \Omega(n^{2-1/m_2(H)})$. This problem requires some background in probability, specifically linearity of expectation.

(a) Let $p \in [0,1]$, and let G be a *random* n-vertex graph obtained by making every pair of vertices adjacent with probability p, independently over all choices. Prove that the expected number of edges in G is $p\binom{n}{2}$.

(b) Prove that for any fixed graph H, the expected number of copies of H in G is at most $p^{e(H)}n^{v(H)}$.

(c) Suppose that H is *2-balanced*, meaning that in the definition of $m_2(H)$, the maximizing subgraph F is H itself. Let X denote the random variable defined as the number of edges of G minus the number of copies of H in G. Prove that if $p = cn^{-1/m_2(H)}$, for some appropriate constant $c > 0$, then $\mathbb{E}[X] \geqslant \Omega(n^{2-1/m_2(H)})$.

(d) Prove that if H is 2-balanced, then $\mathrm{ex}(n,H) \geqslant \Omega(n^{2-1/m_2(H)})$.

(e) Prove that the same conclusion holds even if H is not 2-balanced.

$\star$ 2. In this problem you'll prove the Erdős–Sós conjecture in the special case that T is a path. By the *length* of a path, we mean the number of vertices it has.

$\star\star$ (a) Let G be an n-vertex connected graph with minimum degree $\delta(G)$. Prove that G contains a path of length at least $\min\{n, 2\delta(G)+1\}$.
Hint: Consider a longest path in G, and try to extend it.

(b) Let P_{t+1} denote the path of length $t+1$. Prove that

$$\mathrm{ex}(n, P_{t+1}) \leqslant \left\lceil \frac{(t-1)n}{2} \right\rceil.$$

Hint: Induction on n.

$\star$ (c) Can you characterize the extremal graphs, i.e. the P_{t+1}-free graphs with the maximum number of edges?

3. Provide an alternative proof of Turán's theorem using induction on r. Let G be a K_r-free n-vertex graph.

(a) Let v be a vertex of maximum degree in G. Let A be the set of neighbors of v, and let $B = V(G) \setminus A$.

(b) Form a new graph H by deleting all edges inside B, and adding in all missing edges between A and B. Prove that $e(H) \geqslant e(G)$.

(c) Apply the inductive hypothesis (Turán's theorem for $r-1$) to the induced subgraph on A. Conclude that Turán's theorem holds for r.

Exercises (recommended)

1. Recall that we stated a hypergraph version of the Kővári–Sós–Turán theorem, and proved it (at least in the case $k = 3$) by induction on k. Try proving the $k = 2$ case (i.e. the original Kővári–Sós–Turán theorem) via a similar inductive approach. What does the $k = 1$ case even mean?

2. In class, we only proved the hypergraph Kővári–Sós–Turán theorem in only one special case, namely for $k = 3$ and $s_1 = s_2 = s_3$. Prove the general result, namely that

$$\mathrm{ex}(n, K^{(k)}_{s_1,\dots,s_k}) \leqslant O\left(n^{k-\frac{1}{s_1 s_2 \cdots s_{k-1}}}\right).$$

3. For a k-graph $\mathcal{H}$ and an integer n, let $\pi_n(\mathcal{H}) \coloneqq \mathrm{ex}(n, \mathcal{H})/\binom{n}{k}$.

   $\star$ (a) Prove that $\pi_n(\mathcal{H}) \geqslant \pi_{n+1}(\mathcal{H})$ for all n. Conclude that $\pi(\mathcal{H}) \coloneqq \lim_{n\to\infty} \pi_n(\mathcal{H})$ is well-defined. $\pi(\mathcal{H})$ is called the *Turán density* of $\mathcal{H}$.

   (b) Let H be a graph (i.e. $k = 2$). Find a formula for $\pi(H)$.

4. In this problem, you will prove a supersaturation result for complete bipartite graphs.

   (a) Given two graphs H, G, a *graph homomorphism* from H to G is a function $f : V(H) \to V(G)$ with the property that if uv is an edge of H, then $f(u)f(v)$ is an edge of G. Note that if f is injective, then this yields a copy of H in G. If f is not injective, we say this is a *pseudocopy*.

   Prove that if $v(G) = n$, then there are at most $n^{v(H)}$ homomorphisms from H to G, and at most $\binom{v(H)}{2} n^{v(H)-1}$ pseudocopies of H in G.

   (b) Suppose G has n vertices and $pn^2/2$ edges (we say that G has *edge density* p). Prove[1] that there are at least $p^t n^{1+t}$ homomorphisms from $K_{1,t}$ to G.

   $\star$ (c) Suppose G has n vertices and $pn^2/2$ edges (we say that G has *edge density* p). Prove[2] that there are at least $p^{st} n^{s+t}$ homomorphisms from $K_{s,t}$ to G.

   (d) Deduce from parts (a) and (b) the following supersaturation result. For every $\varepsilon > 0$ and integers s, t, there exists a $\delta > 0$ so that the following holds for sufficiently large n. If G has n vertices and $\varepsilon\binom{n}{2}$ edges, then G has at least $\delta\binom{n}{s+t}$ copies of $K_{s,t}$.

   $\star$ (e) Can you prove analogous results for k-uniform hypergraphs?

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[1] *Hint:* Jensen's inequality.
[2] *Hint:* Use Jensen's inequality again!

Problems (optional)

⋆ 1. Recall that $K_r^{(k)}$ denotes the complete k-uniform hypergraph with r vertices.

(a) Prove[3] that $\mathrm{ex}(n, K_4^{(3)}) \geqslant (\frac{5}{9} + o(1))\binom{n}{3}$.

⋆ (b) Prove that $\mathrm{ex}(n, K_r^{(3)}) \geqslant (1 - (\frac{2}{r-1})^2 + o(1))\binom{n}{3}$ for all $r \geqslant 4$.

(c) Prove that

$$\mathrm{ex}(n, K_r^{(k)}) \leqslant \left(1 - \frac{1}{\binom{r}{k}} + o(1)\right)\binom{n}{k}.$$

⋆⋆ (d) Prove the best known upper bound on $\mathrm{ex}(n, K_r^{(k)})$, namely

$$\mathrm{ex}(n, K_r^{(k)}) \leqslant \left(1 - \frac{1}{\binom{r-1}{k-1}} + o(1)\right)\binom{n}{k}.$$

? (e) Improve any of the bounds above.

2. In this problem, you will study the extremal number of the graph of the 3-dimensional cube, denoted Q_3.

(a) Using Exercise 4, prove that if G is an n-vertex graph with $e(G) \geqslant \Omega(n^{3/2})$, then G has at least $\Omega(p^4 n^4)$ copies of C_4, where $p = 2e(G)/n^2$ is the edge density of G.

(b) Define the *C_4-graph* $C_4(G)$ of G to be the following graph. Its vertices are the *edges* of G, and two such are adjacent in $C_4(G)$ if they are the opposite sides of a C_4 in G. Relate copies of C_4 in $C_4(G)$ to copies of Q_3 in G.
Be careful! Not every C_4 in $C_4(G)$ corresponds to a Q_3 in G; figure out why not.

(c) Prove that $\mathrm{ex}(n, Q_3) \leqslant O(n^{8/5})$.

? (d) Prove a matching lower bound, $\mathrm{ex}(n, Q_3) \geqslant \Omega(n^{8/5})$.

3. In this problem, you will prove a slightly weaker version of Turán's theorem using a technique called *Lagrangians* (or the *Motzkin–Straus inequality*).

(a) Let G be a graph with vertex set $\{v_1, \dots, v_n\}$. Define the *graph polynomial*

$$p_G(x_1, \dots, x_n) = \sum_{v_i v_j \in E(G)} x_i x_j$$

and define the *Lagrangian* $\lambda(G)$ of G to be the maximum of $p_G(x_1, \dots, x_n)$ over all vectors $(x_1, \dots, x_n)$ satisfying $x_i \geqslant 0$ for all i, and $\sum_{i=1}^n x_i = 1$. Prove that this maximum is well-defined.

[3] *Hint:* Split the vertex set into three equal-sized parts.

(b) Prove that $\lambda(G) \geqslant e(G)/n^2$.

$\star$ (c) Let $x = (x_1, \ldots, x_n)$ be a point achieving the maximum in $\lambda(G)$, i.e. a vector with $p_G(x_1, \ldots, x_n) = \lambda(G)$. Moreover, assume that the number of non-zero entries in x is minimized among all such maximizers. Prove that the set of non-zero coordinates in x forms a clique in G.

(d) Deduce from the previous part that if G is K_r-free, then $\lambda(G) \leqslant \frac{1}{2}(1 - \frac{1}{r-1})$.

(e) Conclude that if G is an n-vertex K_r-free graph, then $e(G) \leqslant (1 - \frac{1}{r-1})\frac{n^2}{2}$. Note that this is slightly weaker than the bound in Turán's theorem, but matches it if $r - 1$ divides n.

$\star\star$ 4. In class we proved the following sampling lemma: If G is an n-vertex graph with $e(G) \geqslant \beta\binom{n}{2}$, then the number of m-sets of vertices M with $e(M) \geqslant \alpha\binom{m}{2}$ is at least $(\beta - \alpha)\binom{n}{m}$. In fact, the proof showed that we could replace $\beta - \alpha$ above with $(\beta - \alpha)/(1 - \alpha)$.

Is this result best possible, or close to best possible? That is, is it true that for arbitrarily large n, there exists some n-vertex graph with $e(G) \approx \beta\binom{n}{2}$ and roughly $\frac{\beta-\alpha}{1-\alpha}\binom{n}{m}$ m-sets M with $e(M) \geqslant \alpha\binom{n}{m}$?

For concreteness, feel free to fix your favorite values of α, β, e.g. $\alpha = 1/3$ and $\beta = 2/3$. So can you find a sequence of graphs with around $\frac{2}{3}\binom{n}{2}$ edges so that roughly $\frac{1}{2}\binom{n}{m}$ of the m-sets M satisfy $e(M) \geqslant \frac{1}{3}\binom{m}{2}$?

Exercises (recommended)

⋆1. Recall that the *Turán density* of a k-graph $\mathcal{H}$ is $\pi(\mathcal{H}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{H})/\binom{n}{k}$. Prove the following general form of the supersaturation theorem.

For every k-graph $\mathcal{H}$ and every $\varepsilon > 0$, there exists some $\delta > 0$ so that the following holds for all sufficiently large n. If $\mathcal{G}$ is an n-vertex k-graph with

$$e(\mathcal{G}) \geqslant (\pi(\mathcal{H}) + \varepsilon)\binom{n}{k}$$

then $\mathcal{G}$ has at least $\delta\binom{n}{v(\mathcal{H})}$ copies of $\mathcal{H}$.

2. For a graph H and an integer s, we denote by $H[s]$ the *s-blowup* of H. This is the graph obtained by replacing every vertex of H by an independent set of size s, and replacing every edge of s by a copy of $K_{s,s}$. Similarly, if $\mathcal{H}$ is a k-graph, then $\mathcal{H}[s]$ is the k-graph obtained by replacing every vertex by s vertices, and replacing every edge by a copy of $K^{(k)}_{s,\dots,s}$.

   (a) Check that if $H = K_k$, our two definitions of $K_k[s]$ coincide.

   (b) Deduce from the previous problem the following general form of the Erdős–Stone theorem.

   For every k-graph $\mathcal{H}$ and every positive integer s, we have $\pi(\mathcal{H}[s]) = \pi(\mathcal{H})$.

3. (a) Let $\mathcal{H}, \mathcal{G}$ be k-graphs. A *homomorphism* $\mathcal{H} \to \mathcal{G}$ is a function $V(\mathcal{H}) \to V(\mathcal{G})$ which maps hyperedges to hyperedges. We say that $\mathcal{G}$ is $\mathcal{H}$*-hom-free* if there is no homomorphism $\mathcal{H} \to \mathcal{G}$.

   Let $\mathrm{ex}_{\mathrm{hom}}(n, \mathcal{H})$ denote the maximum number of hyperedges in an $\mathcal{H}$-hom-free n-vertex k-graph. Prove that $\mathrm{ex}_{\mathrm{hom}}(n, \mathcal{H}) \leqslant \mathrm{ex}(n, \mathcal{H})$.

   (b) Now let H be a graph (i.e. let $k = 2$). Determine $\mathrm{ex}_{\mathrm{hom}}(n, H)$.

   (c) Prove that the limit $\pi_{\mathrm{hom}}(\mathcal{H}) := \lim_{n\to\infty} \mathrm{ex}_{\mathrm{hom}}(n, \mathcal{H})/\binom{n}{k}$ exists.

   (d) Prove that $\pi_{\mathrm{hom}}(\mathcal{H}) = \pi(\mathcal{H})$.

4. Prove that if $\mathcal{H}$ is a k-graph, then either $\pi(\mathcal{H}) = 0$ or $\pi(\mathcal{H}) \geqslant k!/k^k$.

Problems (optional)

1. Let G be a graph, and recall that $\alpha(G)$ denotes its independence number.

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(a) By picking a random permutation of $V(G)$, prove that

$$\alpha(G) \geqslant \sum_{v \in V(G)} \frac{1}{\deg(v)+1}.$$

(b) Apply Jensen's inequality to conclude that

$$\alpha(G) \geqslant \frac{n}{d+1},$$

where d is the average degree of G. Recall that you already proved this result, as a consequence of Turán's theorem.

⋆ (c) By more carefully analyzing the proof above, give an alternative proof of Turán's theorem.

⋆ 2. (a) By more carefully analyzing the proof we saw in class, prove the following strengthening of the Erdős–Stone–Simonovits theorem. For every graph H, there exists some $\delta > 0$ such that

$$\mathrm{ex}(n, H) \leqslant t_{\chi(H)-1}(n) + O(n^{2-\delta}).$$

(b) Prove the following converse: for every $\delta > 0$ and every $r \geqslant 2$, there exists a graph H with $\chi(H) = r$ and

$$\mathrm{ex}(n, H) \geqslant t_{r-1}(n) + \Omega(n^{2-\delta}).$$

⋆⋆ 3. In this problem, you will prove the following amazing strengthening of the Kővári–Sós–Turán theorem: if H is a bipartite graph and every vertex on one side has degree at most s, then $\mathrm{ex}(n, H) = O(n^{2-1/s})$.

⋆⋆ (a) Prove[1] the following lemma. For all positive integers a, b, there exists some constant $C > 0$ such that the following holds. Let G be an n-vertex graph with average degree $d \geqslant Cn^{1-1/s}$. Then there exists $U \subseteq V(G)$ with $|U| \geqslant a$ so that every s-tuple of vertices in U has at least $a + b$ common neighbors.

(b) Using the lemma, prove that $\mathrm{ex}(n, H) \leqslant O(n^{2-1/s})$ if every vertex on one side of H has degree at most s.

[1] *Hint:* Pick $x_1, \ldots, x_s$ to be uniformly random vertices of G, chosen with repetition, and let X be the common neighborhood of $x_1, \ldots, x_s$. The desired U can be obtained by deleting some vertices from X, with positive probability.

Exercises (recommended)

1. (a) Prove that if G is an n-vertex K_r-free graph with at least $t_{r-1}(n) - s$ edges, then G can be made $(r-1)$-partite by deleting at most s edges.

   (b) Prove that if G is an n-vertex K_r-free graph with at least $t_{r-1}(n) - s$ edges, then G can be made *complete* $(r-1)$-partite by adding or deleting at most $3s$ edges.

   $\star$ (c) Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds for all sufficiently large n. If G is an n-vertex K_r-free graph with at least $t_{r-1}(n) - \delta n^2$ edges, then G can be turned into $T_{r-1}(n)$ by adding or deleting at most εn^2 edges.

2. On a previous homework, you might have proved the following statement: if an n-vertex directed graph has no copy of a cyclic triangle, then it has at most $\lfloor n^2/2 \rfloor$ edges. The extremal example is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, with all edges oriented in both directions.

   Prove that this extremal problem *does not* exhibit stability. Namely, find another directed graph with $\lfloor n^2/2 \rfloor - o(n^2)$ edges and no cyclic triangle, which cannot be turned into the extremal example above by adding/deleting $o(n^2)$ edges.

3. In this problem you'll prove lower bounds for the extremal numbers of cycles.

   (a) Let p be a prime, $2 \leqslant \ell \leqslant p$ a positive integer, and let $a_1, \dots, a_\ell$ be ℓ distinct elements of $\mathbb{F}_p$. Prove that the vectors

   $$(1, a_1, a_1^2, \dots, a_1^{\ell-1}), \qquad (1, a_2, a_2^2, \dots, a_2^{\ell-1}), \qquad \dots \qquad (1, a_\ell, a_\ell^2, \dots, a_\ell^{\ell-1})$$

   are linearly independent in $\mathbb{F}_p^\ell$.

   (b) Let p and ℓ be as above, and consider the following bipartite graph G. Its two parts are X and Y, where $X = \mathbb{F}_p^\ell$ and Y consists of all lines in $\mathbb{F}_p^\ell$ of the form

   $$\{(b_1, \dots, b_\ell) + t \cdot (1, a, a^2, \dots, a^{\ell-1}) : t \in \mathbb{F}_p\}.$$

   Make $x \in X$ and $y \in Y$ adjacent in G if and only if the point x lies on the line y. Prove that G has $n = 2p^\ell$ vertices and $p^{\ell+1} = \Theta(n^{1+1/\ell})$ edges.

   $\star$ (c) Prove that if $\ell \in \{2, 3, 5\}$, then G is $C_{2\ell}$-free. Conclude that $\mathrm{ex}(n, C_{2\ell}) = \Theta(n^{1+1/\ell})$.

   (d) What goes wrong if $\ell \notin \{2, 3, 5\}$?

   ? (e) Modify this construction to work for $\ell = 7$.

✣ 4. Recall that the *distance* between two vertices u, v in a graph G, denoted $d_G(u, v)$, is the number of edges in the shortest path connecting them.

$\star$ means that a problem is hard.
? means that a problem is open.
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(a) Prove that if H is a spanning subgraph of G (i.e. $V(H) = V(G)$ and $E(H) \subseteq E(G)$), then $d_G(u, v) \leqslant d_H(u, v)$ for all u, v.

(b) Given an integer k, a *k-spanner* of G is a subgraph $H \subseteq G$ for which

$$d_G(u, v) \leqslant d_H(u, v) \leqslant k \cdot d_G(u, v)$$

for all u, v. Prove[1] that every n-vertex graph G, regardless of how many edges it has, contains a $(2\ell - 1)$-spanner H with $e(H) \leqslant O(n^{1+1/\ell})$, for any $\ell \geqslant 1$.

Remark: Spanners are very important in computer science, as they allow us to approximate distances in G while using much less storage than it would take to store all of G. For example, even if G has $\Theta(n^2)$ edges, the result above shows that we can approximate distances in G up to a factor of 100 by storing only $O(n^{1.02})$ edges.

(c) Prove that this result is tight if $\ell \in \{2, 3, 5\}$. That is, there exists an n-vertex graph G containing no $(2\ell - 1)$-spanner with fewer than $cn^{1+1/\ell}$ edges, for some constant $c > 0$.

Problems (optional)

1. In this problem you'll see some variants of the supersaturation theorem for triangles.

   (a) Prove that if an n-vertex graph has $\lfloor n^2/4 \rfloor + 1$ edges, then it contains at least $\lfloor n/2 \rfloor$ triangles.

   (b) Prove that this bound is tight.

   ⋆⋆(c) Prove that if an n-vertex graph has $\lfloor n^2/4 \rfloor + 1$ edges, then it contains at least $\lfloor n/6 \rfloor$ triangles all sharing a single edge.

   ⋆(d) Prove that this bound is tight.

⋆2. Remove the minimum degree assumption from from the proof of Proposition 11.3, thus proving that $\mathrm{ex}(n, C_5) = \lfloor n^2/4 \rfloor$ for all sufficiently large n.

⋆⋆3. Prove the following general stability theorem: for every H and every $\varepsilon > 0$, there exists $\delta > 0$ so that the following holds for all sufficiently large n. If G is an n-vertex K_r-free graph with at least $t_{\chi(H)-1}(n) - \delta n^2$ edges, then G can be made $(\chi(H) - 1)$-partite by deleting at most εn^2 edges.

⋆⋆4. Prove the following combination of the supersaturation and stability theorems. For every $r \geqslant 3$ and every $\varepsilon > 0$, there exist $\delta, \gamma > 0$ such that the following holds for all sufficiently large n. If G is an n-vertex graph with at most γn^r copies of K_r and minimum degree at least $(1 - \frac{1}{r-1} - \delta)n$, then G can be made $(r-1)$-partite by deleting at most εn^2 edges.

[1] *Hint:* Greedily add edges to H while not creating a short cycle.

5. Let $\mathcal{F}$ be a finite collection of bipartite graphs, none of which is a forest. A famous conjecture of Erdős and Simonovits, called the *compactness conjecture*, asserts that there exists some $H \in \mathcal{F}$ such that

$$\mathrm{ex}(n, \mathcal{F}) \leqslant \mathrm{ex}(n, H) \leqslant C \cdot \mathrm{ex}(n, \mathcal{F}),$$

where $C > 0$ is an absolute constant, depending only on $\mathcal{F}$.

(a) Prove that the first inequality above holds for any $H \in \mathcal{F}$.

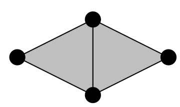
$\star$ (b) Prove that the compactness conjecture can be false if we allow $\mathcal{F}$ to be infinite.

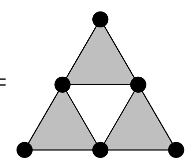
$\star$ (c) Prove that the compactness conjecture can be false if we allow $\mathcal{F}$ to contain forests.

? (d) Prove or disprove the compactness conjecture.

$\star\star$ (e) The compactness conjecture is known to be false for hypergraphs! You'll see this in this part and the next.

Consider the following two 3-partite 3-graphs:

$K^{(3)}_{1,1,2} =$ 

$T =$ 

Prove that $\mathrm{ex}(n, K^{(3)}_{1,1,2}) = \Theta(n^2)$ and $\mathrm{ex}(n, T) = \Theta(n^2)$.

$\star\star\star\star\star$ (f) Prove that $\mathrm{ex}(n, \{K^{(3)}_{1,1,2}, T\}) = o(n^2)$, thus disproving the compactness conjecture for hypergraphs.

Exercises (recommended)

1. (a) Prove that $r(3,3) = 6$.
   (b) Prove that $r(3,4) = 9$.
   (c) Prove that $r(4,4) \leqslant 18$.
   $\star$ (d) Prove that $r(4,4) = 18$.
   ? (e) The best known bounds on $r(5,5)$ are $43 \leqslant r(5,5) \leqslant 46$. Can you improve either of these bounds?
   $\star$ (f) Prove that $r(3;3) = 17$ (recall that $r(3;3)$ is the 3-color Ramsey number).

2. (a) Prove that
$$r(3;q) \leqslant 1 + q!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right).$$
   (b) Conclude that $r(3;q) \leqslant \lceil e \cdot q! \rceil$, where e is Euler's constant.

3. (a) Using the fact that $r(k) < 4^k$, prove that $r(k;q) < 4^{4^{4^{\cdot^{\cdot^{\cdot^{k}}}}}}$, where the number of 4s is $\lceil \log_2 q \rceil$.
   (b) Prove Theorem 13.5 in the notes. In particular, derive the bound $r(k;q) < q^{qk}$, which is much stronger than that in part (a).

4. Prove the following supersaturation version of Ramsey's theorem, which is usually called a *Ramsey multiplicity* result.

   For all positive integers k, q, there exists some $\delta > 0$ so that the following holds for every sufficiently large N. No matter how we q-color the edges of K_N, there are at least $\delta\binom{N}{k}$ monochromatic copies of K_k.

Problems (optional)

✣ 1. (a) Prove that for any positive integer q, there exists a positive integer $N = N(q)$ such that the following holds. For any q-coloring of $[\![N]\!]$, there exist $x, y, z \in [\![N]\!]$ such that $x, y, z, x+y, y+z, x+y+z$ all receive the same color. (Note that $x+z$ is omitted!)
   (b) Generalize the previous part as follows. Prove that for all positive integers q, t, there exists a positive integer $N = N(q,t)$ such that the following holds. For any q-coloring of $[\![N]\!]$, there exist $x_1, \ldots, x_t \in [\![N]\!]$ such that the sums $\sum_{i=a}^{b} x_i$ all receive the same color, for all non-empty $1 \leqslant a \leqslant b \leqslant t$.

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⋆(c) Prove that in part (b), one can moreover ensure that the numbers $x_1, \ldots, x_t$ are all distinct.

2. Let H be a graph. The 2-*colored extremal number* of H, denoted $\mathrm{ex}_2(n, H)$, is defined to be the maximum number of edges in an n-vertex graph G for which there exists a 2-coloring of $E(G)$ containing no monochromatic copy of H.

   Find an exact formula for $\mathrm{ex}_2(n, K_k)$.

⋆3. Let $f, g_1, \ldots, g_q : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that there exist $\varepsilon, \delta > 0$ such that whenever $x, y \in \mathbb{R}$ satisfy $f(x) - f(y) \geqslant \varepsilon$, then

$$\max_{i \in [\![q]\!]} (g_i(x) - g_i(y)) \geqslant \delta.$$

   Prove that if $g_1, \ldots, g_q$ are all bounded, then f is bounded as well.

✣⋆4. In class, we proved that $r(k) < 4^k$ using the Erdős–Szekeres argument. Ramsey's original proof used a *different* argument, which yielded the worse bound $r(k) \leqslant k!$. Find a natural argument yielding this bound. (That is, don't simply quote or rederive the Erdős–Szekeres argument!)

Exercises (recommended)

1. Prove that the 3-uniform Ramsey number satisfies

$$r_3(k) \geqslant 2^{ck^2}$$

for some absolute constant $c > 0$.

2. In a red/blue coloring of $E(K_N)$, denote by $\deg_R(v), \deg_B(v)$ the red and blue degrees, respectively, of a vertex v.

   (a) Prove that the number of monochromatic triangles in such a coloring is equal to

$$\frac{1}{2}\left(\sum_{v \in V(K_N)} \left[\binom{\deg_R(v)}{2} + \binom{\deg_B(v)}{2}\right] - \binom{N}{3}\right).$$

   (b) Prove that every 2-coloring of $E(K_6)$ contains at least two monochromatic triangles, and in particular obtain a new proof that $r(3) \leqslant 6$.

   ⋆(c) Prove that every 2-coloring of $E(K_N)$ contains at least $\frac{N(N-1)(N-5)}{24}$ monochromatic triangles.

3. On yesterday's homework, you proved a supersaturation version of Ramsey's theorem. In the 2-color case, it states that for every $k \geqslant 3$, there is some $\delta > 0$ such that for every sufficiently large N, every 2-coloring of $E(K_N)$ contains at least $\delta\binom{N}{k}$ monochromatic copies of K_k.

   The *Ramsey multiplicity constant* of K_k, denoted $c(K_k)$, is defined to be the supremum of all δ for which this statement is true.

   (a) Prove that $c(K_k) \geqslant \binom{r(k)}{k}^{-1}$. Conclude that $c(K_k) \geqslant 4^{-k^2}$.

   (b) Prove that $c(K_k) \leqslant 2^{1-\binom{k}{2}}$.

   (c) Prove[1] that $c(K_3) = \frac{1}{4}$.

4. For every $N \geqslant 3$, give N points in $\mathbb{R}^2$ with no three in convex position. This shows that the assumption in Theorem 14.5 that no three points are collinear is necessary.

5. Prove that for every $k \geqslant 3$, there exists some N so that the following holds. Among any N points in the plane, there are either k points lying on a line, or k points in convex position.

[1] *Hint:* Use the previous exercise.

6. Let $N = r_3(k)$, and let $p_1, \ldots, p_N$ be points in $\mathbb{R}^2$ with no three collinear. Define $\chi : E(K_N^{(3)}) \to \{\text{even}, \text{odd}\}$ by

$$\chi(\{i,j,\ell\}) \coloneqq \begin{cases} \text{even} & \text{if there are an even number of points } p_m \text{ in the triangle } p_i p_j p_\ell, \\ \text{odd} & \text{otherwise.} \end{cases}$$

Prove that a monochromatic $K_k^{(3)}$ under χ corresponds to k points in convex position. Conclude that $\mathrm{Kl}(k) \leqslant r_3(k)$, and in particular obtain a new proof of Theorem 14.5.

Problems (optional)

1. (a) By more carefully analyzing the proof of Theorem 13.7 in the notes, prove that

$$r(k) > \left(\frac{1}{e\sqrt{2}} - o(1)\right) k 2^{k/2}.$$

⋆ (b) Improve this bound by a constant factor.

? (c) Improve this bound by a super-constant factor.

⋆ 2. Prove that $\mathrm{Kl}(5) = 9$.

3. Given two graphs G, H, their *lexicographic product* $G \cdot H$ is defined as follows. Its vertex set is $V(G \cdot H) = V(G) \times V(H)$, and two vertices $(a,b), (c,d)$ are adjacent if either $ac \in E(G)$ or $a = c$ and $bd \in E(H)$.

   (a) Compute the size of the largest clique and the largest independent set in $G \cdot H$.

   (b) Prove that the Ramsey number $r(k)$ satisfies $r(k+1) > k^{\log_2(5)}$ for all k that are powers of 2.

   [Note that this already disproves Turán's belief that $r(k)$ may grow only quadratically as a function of k.]

   ⋆⋆ (c) Using the same approach, find an *explicit* construction of a coloring witnessing that $r(k)$ grows super-polynomially in k. In other words, for any $C > 0$ and any sufficiently large k, find an explicit 2-coloring of $E(K_N)$, where $N = k^C$, with no monochromatic clique of order k.

   ? (d) Can you use such an approach to resolve Open problem 13.8?

4. A collection of points in $\mathbb{R}^d$ is said to be *in general position* if no $d+1$ of them lie on a $(d-1)$-dimensional hyperplane. (So in two dimensions, this says that no three are collinear, in three dimensions it says that no four are coplanar, etc.)

   (a) Prove that among any $d+3$ points in $\mathbb{R}^d$ which are in general position, there are $d+2$ in convex position.

(b) Given $k \geqslant d+2$, let $N = r_{d+2}(d+3, k)$. Prove that among any N points in $\mathbb{R}^d$ in general position, there are k in convex position.

(c) Prove that among any $\mathrm{Kl}(k)$ points in $\mathbb{R}^d$, no three collinear, there are k in convex position.

[This is stronger than the result in (b) in two ways: the bound is *independent* of d, and the assumption is weakened from general position to no three collinear.]

✣ 5. A *subdivision* of a graph H is obtained from H by replacing every edge of H by a path of some length (not necessarily the same length for all edges, and paths of length 1 are allowed, so that H is a subdivision of itself). A famous conjecture of Hajós asserts that if $\chi(G) \geqslant k$, then G contains a subdivision of K_k as a subgraph.

(a) Prove that Hajós' conjecture is true for $k \leqslant 3$.

⋆ (b) Prove that Hajós' conjecture is true for $k = 4$.

(c) Prove that Hajós' conjecture for $k = 5$ implies the four-color theorem. Conclude that it is probably pretty hard to prove the $k = 5$ case.

(d) Prove that if Hajós' conjecture is true, then $r(k) \leqslant 3k^3$. Conclude that Hajós' conjecture is false.

✣ 6. A classical fact in graph theory is that there exist triangle-free graphs of arbitrarily high chromatic number. In this exercise, you will see a non-standard Ramsey-theoretic proof.

For an integer N, let S_N be a graph with vertex set $\binom{[\![N]\!]}{2}$, where we think of the vertices of S_N as ordered pairs (a, b) with $1 \leqslant a < b \leqslant N$. The edges of S_N consist of all pairs of the form $((a, b), (b, c))$ for $a < b < c$.

(a) Prove that S_N is triangle-free.

(b) Prove that $\chi(S_N) \to \infty$ as $N \to \infty$.

✣ 7. (a) Let $K_{\mathbb{N}}$ denote the complete graph whose vertex set is $\mathbb{N}$. Prove the "infinite Ramsey theorem": for any positive integer q, and any q-coloring of $K_{\mathbb{N}}$, there is an infinite monochromatic clique.

(b) State and prove the infinite *hypergraph* Ramsey theorem.

✣ 8. Prove that there is an infinite set $S \subseteq \mathbb{N}$ such that for every $a, b \in S$, the number $a + b$ has an even number of prime factors (counted without multiplicity).

Exercises (recommended)

1. Prove that $r(K_{1,k}) = 2k$ if k is odd, and $r(K_{1,k}) = 2k - 1$ if k is even.
2. Let kK_2 denote a matching with k edges, that is, a disjoint union of k copies of the single-edge graph K_2. Prove that $r(kK_2) = 3k - 1$ for all $k \geqslant 1$.
3. (a) Prove that $r(T; q) \leqslant O(qn)$ for every $q \geqslant 2$ and every n-vertex tree T.

   $\star$ (b) Prove that $r(T; q) = \Theta(qn)$ for every $q \geqslant 2$ and every n-vertex tree T.
4. Prove that every non-empty forest has degeneracy 1.
5. Prove[1] that there exist absolute constants $C, c > 0$ such that the following holds for all n. There exists an n-vertex graph H with degeneracy $d \geqslant c \log_2 n$ and $r(H) \leqslant Cn$.

   Note that this result is close to optimal; by Theorem 15.8, such an upper bound on $r(H)$ cannot hold if $c > 2$.

Problems (optional)

1. Prove that for every integer k and for every n-vertex tree T, we have

$$r(K_k, T) = (k-1)(n-1) + 1.$$

$\star\star$ 2. Let P_k denote a k-vertex path. Prove that for all $k \geqslant \ell \geqslant 2$,

$$r(P_k, P_\ell) = k + \left\lfloor \frac{\ell}{2} \right\rfloor - 1.$$

3. (a) Prove that

$$r(C_{2k+1}; q) > 2^q k$$

   for all $k \geqslant 1, q \geqslant 2$.

   $\star$ (b) Prove that

$$r(C_{2k+1}; q) \leqslant C(q+2)!k,$$

   for some absolute constant C.

   ? (c) The previous two parts show that $r(C_{2k+1}; q)$ grows linearly in k and between exponentially and super-exponentially in q. Determine whether the true behavior is exponential or super-exponential.

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[1] *Hint:* Use a lot of isolated vertices.

⋆⋆4. Prove that $r(K_{k,k}) \leqslant O(2^k \log k)$.

5. For a bipartite graph H and a number $\delta > 0$, let $r_d(H;\delta)$ denote the minimum integer N such that every N-vertex graph with at least $\delta\binom{N}{2}$ edges has a copy of H.

   (a) Using what you know about extremal numbers of bipartite graphs, prove that $r_d(H;\delta)$ is well-defined, i.e. that this number is finite for all bipartite H and all $\delta > 0$.

   (b) By more carefully examining your solution to the previous part, show that for every bipartite graph H, there exists some $C > 0$ such that

$$r_d(H;\delta) \leqslant \left(\frac{1}{\delta}\right)^C$$

   for all $0 < \delta \leqslant \frac{1}{2}$.

   (c) Let H be a graph, and suppose G is an N-vertex graph with $\delta\binom{N}{2}$ edges and with no copy of H. Prove[2] that if q is an integer satisfying $(1-\delta)^q\binom{N}{2} < 1$, then

$$r(H;q) > N.$$

   (d) Fix a bipartite graph H, and let C be the constant from part (b). Using the previous parts, prove that

$$r_d\left(H;\frac{2C\ln q}{q}\right) \leqslant r(H;q) \leqslant r_d\left(H;\frac{1}{q}\right),$$

   This shows that $r(H;q)$ and $r_d(H;1/q)$ are closely related for bipartite H. In particular, we see that Ramsey numbers of bipartite graphs are essentially controlled by extremal graph theory.

[2] *Hint:* Randomly permute the vertices of G to obtain q copies $G_1,\dots,G_q$. Show that with positive probability, every edge of K_N appears in at least one G_i.

Exercises (recommended)

1. (a) Prove that every 2-coloring of $E(K_N)$ contains a monochromatic N-vertex tree.

   (b) Prove that for every $q \geqslant 2$, there exists some $\delta > 0$ such that the following holds. In any q-coloring of $E(K_N)$, one of the color classes contains *all* of the trees on δN vertices.

2. (a) Let G, H be graphs such that H is connected. Prove that

$$r(G, H) \geqslant (\chi(G) - 1)(|H| - 1) + 1,$$

   where $|H|$ denotes the number of vertices of H.

   (b) Let $\sigma(G)$ denote the minimum number of vertices that can appear in a color class among all proper $\chi(G)$-colorings of G. Strengthen the result above to

$$r(G, H) \geqslant (\chi(G) - 1)(|H| - 1) + \sigma(G).$$

3. Let ℓK_2 denote the *matching graph*, consisting of 2ℓ vertices and ℓ disjoint edges. Prove that $r(\ell K_2, K_k) = 2\ell + k - 2$ for all integers $\ell \geqslant 1, k \geqslant 2$.

4. (a) Let $k, \ell \geqslant 2$. Prove that in any sequence of $(k-1)(\ell-1)+1$ distinct real numbers, there is an increasing subsequence of length k or a decreasing subsequence of length ℓ.

   (b) Prove that the result in (a) is best possible, by finding a sequence of $(k-1)(\ell-1)$ distinct real numbers with no increasing subsequence of length k and no decreasing subsequence of length ℓ.

5. Prove that any sequence of (not necessarily distinct) real numbers of length $(k-1)^3+1$ contains a subsequence of length k that is strictly increasing, strictly decreasing, or constant. Prove that this bound is best possible.

Problems (optional)

✣ 1. (a) Prove that any infinite sequence of distinct real numbers contains an infinite subsequence that is (non-strictly) increasing or (non-strictly) decreasing.

   (b) Prove the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence.

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2. Let $v_1, \ldots, v_N \in \mathbb{R}^d$ be vectors, and let $(v_i)_j$ denote the jth coordinate of v_i, for any $j \in [\![d]\!]$. Prove that if $N \geqslant (k-1)^{2^d} + 1$, then there is a *totally monotone* subsequence of length k; that is, there are $i_1 < \cdots < i_k$ such that for all $j \in [\![d]\!]$, we have $(v_{i_1})_j \geqslant \cdots \geqslant (v_{i_k})_j$ or $(v_{i_1})_j \leqslant \cdots \leqslant (v_{i_k})_j$.

⋆3. Prove[1] that for every $\Delta \geqslant 2$, there exists $C_\Delta > 0$ such that the following holds for every n. If $N \geqslant C_\Delta n$, then in any two-coloring of $E(K_N)$, one of the color classes contains *all* n-vertex graphs of maximum degree at most Δ.

✣4. Prove that a graph has degeneracy at most d if and only if its vertices can be ordered as $v_1, \ldots, v_n$ such that v_i has at most d neighbors preceding it in the order.

Remark: This alternative definition is very useful when trying to prove things like the Burr–Erdős conjecture, as it suggests a good order in which to try to embed the vertices one by one.

⋆⋆5. (a) Formalize the proof sketch we saw in class, and prove that if H is an n-vertex graph with maximum degree Δ, then

$$r(H) \leqslant 2^{C\Delta(\log \Delta)^2} n,$$

where C is an absolute constant.

⋆⋆(b) Improve this bound to

$$r(H) \leqslant 2^{C\Delta \log \Delta} n.$$

?(c) Improve this bound to

$$r(H) \leqslant 2^{C\Delta} n.$$

⋆6. Let kK_3 denote the graph that is the disjoint union of k triangles. Prove[2] that for all $k \geqslant 2$, we have $r(kK_3) = 5k$.

7. Prove that $r(C_4; q) \leqslant q^2 + q + 2$ for all $q \geqslant 2$.

8. Erdős conjectured that if $\chi(H) = k$, then $r(H) \geqslant r(K_k)$.

(a) Prove this conjecture for $k \leqslant 3$.

⋆⋆(b) Find a counterexample to this conjecture for $k = 4$.

✣9. Prove that if N is sufficiently large, then the following holds. Among any N points in the plane, there are three of them that determine an angle greater than $179°$.

⋆⋆10. Construct[3], for every $k \geqslant 4$, a collection of 2^{k-2} points in $\mathbb{R}^2$, no three collinear, with no k of them in convex position. Deduce that $\mathrm{Kl}(k) \geqslant 2^{k-2} + 1$.

[1] *Hint:* There is a black-box reduction to the statement of Theorem 15.12.
[2] *Hint*: Induct on k. The base case is super annoying, but the inductive step is nice.
[3] *Hint:* A solution for $k = 5$ is given on the next page; try to generalize it.

Exercises (recommended)

1. (a) Prove that for every forest F, there exists some integer N such that the following holds. In any coloring of $E(K_N)$, with an arbitrary number of colors, there is a monochroamtic or rainbow copy of F.

   (b) Prove that the result of part (a) is false for any graph H which is not a forest.

2. Let us say that a coloring of $E(K_k)$ is *semi-starry* if the vertices can be sorted as $v_1, \ldots, v_k$ such that all edges $v_i v_j$, where $j > i$, are of the same color. (The only difference from a starry coloring is that we do not require these colors to be distinct.)

   (a) Prove that if $N \geqslant (k-1)^2 + 1$, then any semi-starry coloring of $E(K_N)$ contains a monochromatic or starry K_k. Such a result was implicitly used in the proof of Theorem 16.3.

   (b) Prove that if $N \geqslant k^{4k}$, then any coloring of $E(K_N)$, with an arbitrary number of colors, contains a rainbow or a semi-starry K_k.

   ⋆ (c) Show that, for some absolute constant $c > 0$, there exists a coloring of $E(K_N)$, where $N = k^{ck}$, with no rainbow or semi-starry K_k. Thus, the result of part (b) is best possible up to the constant factor in the exponent.

3. Usually, the canonical Ramsey theorem is stated in an ordered version. Here, we label $V(K_N)$ as $v_1, \ldots, v_N$, and we say that indices $i_1 < \cdots < i_k$ form a *left-starry* K_k if all edges from v_{i_j} to v_{i_ℓ} receive the same color, for all $\ell > j$, and these colors are distinct for different j. Similarly, it's *right-starry* if the same holds for all $\ell < j$.

   Prove that for every k, there exists some N such that any coloring of $E(K_N)$, with an arbitrary number of colors and with the fixed vertex labeling $v_1, \ldots, v_N$, there is a monochromatic, rainbow, left-starry, or right-starry K_k.

Problems (optional)

1. Prove the bipartite canonical Ramsey theorem, which states the following. For every $k \geqslant 2$, there exists some N such that in any coloring of $E(K_{N,N})$, with an arbitrary number of colors, there is a $K_{k,k}$ which is monochromatic, rainbow, or starry.

   (Here, a $K_{k,k}$ is *rainbow* if all k^2 edges receive different colors, and is *starry* if it is colored by exactly k distinct colors, each of whose color classes is a star $K_{1,k}$.)

✣ 2. Fekete's lemma is an important result in real analysis. It says that if a function $f : \mathbb{N} \to \mathbb{R}$ is *supermultiplicative*, meaning that $f(m+n) \geqslant f(m)f(n)$ for all m, n, then the limit $\lim_{n\to\infty} f(n)^{1/n}$ exists.

⋆ means that a problem is hard.

? means that a problem is open.

✣ means that a problem is on a topic beyond the scope of the course.

$\star$ (a) Prove Fekete's lemma (or take it as a given and move on).

(b) Prove that, for any fixed $k \geqslant 3$, the limit

$$\lim_{q\to\infty} r(k;q)^{1/q}$$

exists.

? (c) Prove that, for any fixed $q \geqslant 2$, the limit

$$\lim_{k\to\infty} r(k;q)^{1/k}$$

exists.

$\star$ 3. There is also a canonical Ramsey theorem for hypergraphs.

(a) Find a list of colorings of $K_N^{(3)}$ that are canonical, in the sense that every subset of vertices is colored in the same way. Monochromatic and rainbow are obvious examples, but what are the correct hypergraph notions of starry?

$\star$ (b) Prove the canonical Ramsey theorem for 3-uniform hypergraphs: for every k, there exists some N such that in any coloring of $E(K_N^{(3)})$, with an arbitrary number of colors, there is a copy of $K_k^{(3)}$ that is colored according to one of the canonical colorings in the list you found.

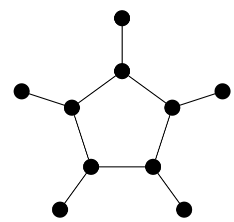
$\star\star$ (c) Extend these results to t-uniform hypergraphs.

Exercises (recommended)

1. Let P_k denote the path graph with k vertices and $k-1$ edges.

   (a) Prove that, for any $q \geqslant 2$, the star $K_{1,q+1}$ is q-color Ramsey for P_3.

   (b) Prove that the following graph, obtained by adding a leaf to each vertex of C_5, is 2-color Ramsey for P_4.



   (c) Give an example of a graph that is 2-color Ramsey for P_5.

2. Prove that for every $n, q \geqslant 2$, there exists some N such that $K_{N,N}$ is q-color Ramsey for $K_{n,n}$.

3. A graph G is *minimally Ramsey* for H if G is Ramsey for H, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H. H is called *Ramsey finite* if there are only a finite number of minimally Ramsey graphs for H, and *Ramsey infinite* otherwise.

   (a) Let $G = K_3 * C_\ell$, where $\ell \geqslant 3$ is odd. Prove that G is minimally Ramsey for K_3. Conclude that K_3 is Ramsey infinite.

   (b) Determine the set of Ramsey minimal graphs for $K_{1,2}$.

   ⋆(c) Prove that $K_{1,k}$ is Ramsey finite if and only if k is odd.

4. A graph G is called *q-minimally Ramsey* for a graph H if G is Ramsey for H in q colors, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H in q colors.

   (a) Prove that if G is q-minimally Ramsey for H, then every edge of G lies in at least q copies of H.

   (b) Prove that if G is q-minimally Ramsey for H, then G has at least $q^{e(H)-1}$ copies of H.

   (c) Prove Proposition 17.9 from the notes.

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? means that a problem is open.
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Problems (optional)

1. (a) Prove that if G is Ramsey for K_k, then $\chi(G) \geqslant r(k)$.
   (b) Prove that if G is Ramsey for K_k, then $e(G) \geqslant \binom{r(k)}{2}$. That is, the complete graph $K_{r(k)}$ has the fewest number of edges among all graphs Ramsey for K_k.
   (c) Find an example of a graph H for which there is a graph G which is Ramsey for it, but G has fewer edges than $\binom{r(H)}{2}$. That is, it was really important that we took $H = K_k$ above.

⋆2. Recall the definitions from exercise 3. In this problem, you'll prove that every tree which is not a star is Ramsey infinite.
   (a) Prove that if T is an n-vertex tree, then every graph G of chromatic number at least $n^2 + 1$ is Ramsey for G.
   (b) Prove that if T is a tree which is not a star, then for any forest F, F is not Ramsey for T.
   ⋆(c) Prove[1] that for every $k, g \geqslant 3$, there is a graph G with chromatic number at least k and girth at least g (the *girth* of G is the length of the shortest cycle in G).
   (d) Using the previous parts, prove that for every tree T which is not a star, and for every integer g, there is a graph G on at least g vertices which is minimally Ramsey for T. Conclude that T is Ramsey infinite.

3. (a) Prove that for any $\ell \geqslant 4$, the cycle C_ℓ is a subgraph of a triangle tree (and hence Ramsey obligatory for K_3).
   (b) Prove that K_4 is not a subgraph of any triangle tree.

✣4. (a) Prove that if G is Ramsey for H and H is Ramsey for F, then G is 4-color Ramsey for F.
   (b) Find an example[2] of G, H, F as above for which G is *not* 5-color Ramsey for F.

5. Let G be a graph. Recall that the s-*blowup* of G, denoted $G[s]$, is the graph obtained by replacing each vertex of G by s vertices, and replacing each edge of G by a complete bipartite graph $K_{s,s}$.
   (a) Prove that for every $s \geqslant 2$, there exists some $N = N(s)$ such that $K_6[N]$ is Ramsey for $K_3[s]$.
   (b) Prove that $N(s) > 2^s$ for all $s \geqslant 4$.
   ⋆(c) Prove the following generalization of part (a): For every graph H and every integer s, if G is Ramsey for H, then there exists N such that $G[N]$ is Ramsey for $H[s]$.

[1] Hint: Consider a random n-vertex graph where each edge is included with probability $p = n^{-1/(2g)}$.
[2] *Hint:* $H = K_3$.

✣★★6. Prove[3] Theorem 17.3 from Theorem 17.5. You should in fact assume the following strengthening of Theorem 17.5; in the same setup as in the theorem statement, we have that

$$\Pr(G \text{ is Ramsey for } H \text{ in } q \text{ colors}) \begin{cases} \geqslant 1 - e^{-cpN^2} & \text{if } p \geqslant CN^{-1/m_2(H)}, \\ \leqslant e^{-cpN^2} & \text{if } p \leqslant cN^{-1/m_2(H)}. \end{cases}$$

[3] *Hint:* Use Harris's inequality/FKG inequality (and look it up if you've never heard of it).