Exercises (recommended)

1. Prove, as claimed in class, that $\frac{2^n-1}{n} \leqslant \text{magic}(n) < 2^{n-1}$ for all integers n.
2. What does Turán's theorem mean in case $r = 2$? Is the theorem true in that case?
3. Find a general formula for $t_{r-1}(n)$, in terms of n, r, and $s \coloneqq n \pmod{r-1}$.
4. Prove that $T_{r-1}(n)$ maximizes number of edges among all complete $(r-1)$-partite graphs (that is, that any complete $(r-1)$-partite graph with parts of sizes *different* from $\lfloor n/(r-1) \rfloor$ or $\lceil n/(r-1) \rceil$ has fewer edges than $T_{r-1}(n)$).
5. Provide an alternative proof of Turán's theorem by induction on n (with inductive steps of size 1) by deleting a vertex of minimum degree.
6. Let G be an n-vertex graph. Recall that the *independence number* of G, denoted $\alpha(G)$, is the size of the largest set of vertices in G containing no edge. Let Δ be the maximum degree of G, and let d be the average degree of G.
   (a) Prove that $\chi(G) \leqslant \Delta(G) + 1$. Conclude that $\alpha(G) \geqslant n/(\Delta + 1)$.
   (b) Using Turán's theorem, prove that $\alpha(G) \geqslant n/(d+1)$. Note that this is a (much!) stronger result.

Problems (optional)

1. In this problem, you will show that $\text{magic}(n) = \Omega(2^n/\sqrt{n})$, following the argument of Erdős and Moser. This problem assumes some familiarity with probability, specifically Chebyshev's inequality.
   (a) Let $a_1, \ldots, a_n \in [\![M]\!]$, and suppose that all subsets have distinct sums. Let $\xi_1, \ldots, \xi_n$ be independent random variables, each taking on the values 0 or 1 with probability $\frac{1}{2}$, and let
   $$X = \sum_{i=1}^{n} \xi_i a_i.$$
   Prove that for every integer x, we have that $\Pr(X = x) \leqslant 2^{-n}$.
   (b) Prove that $\text{Var}(X) \leqslant M^2 n/4$.
   (c) Let $\lambda > 1$ be some parameter. Using Chebyshev's inequality, plus the previous two parts, prove that
   $$1 - \frac{1}{\lambda^2} \leqslant \Pr\left(|X - \mathbb{E}[X]| < \frac{\lambda M \sqrt{n}}{2}\right) \leqslant 2^{-n}\left(\lambda M \sqrt{n} + 1\right).$$

$\star$ means that a problem is hard.
? means that a problem is open.
✢ means that a problem is on a topic beyond the scope of the course.

(d) By picking λ appropriately, prove that $M = \Omega(2^n/\sqrt{n})$. Deduce that $\text{magic}(n) = \Omega(2^n/\sqrt{n})$. What is the best constant factor you can obtain by optimizing λ?

✣ 2. (a) Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^d$ with $\|v_i\| \geqslant 1$ for all i, where $\| \cdot \|$ denotes the usual Euclidean length of a vector. Prove that there are at most $\lfloor \frac{n^2}{4} \rfloor$ pairs v_i, v_j with $\|v_i + v_j\| < 1$.

⋆ (b) Fix a probability distribution on $\mathbb{R}^d$, and let X, Y be two independent random vectors drawn according to this distribution. Prove that

$$\Pr(\|X + Y\| \geqslant 1) \geqslant \frac{1}{2} \Pr(\|X\| \geqslant 1)^2.$$

(c) Find a probability distribution on $\mathbb{R}^d$ for which the above bound is tight.

3. A *directed graph* is a graph in which every edge is assigned one of the two possible directions. In a directed graph, we allow *anti-parallel edges*, i.e. $x \to y$ and $y \to x$ may both be edges in the same directed graph. An *oriented graph* is a directed graph without anti-parallel edges.

   (a) A *cyclic triangle* is the oriented graph on 3 vertices with edges $x \to y, y \to z, z \to x$. What is the maximum number of edges in an n-vertex oriented graph without a cyclic triangle?

   (b) A *transitive triangle* is the oriented graph on 3 vertices with edges $x \to y, y \to z, x \to z$. What is the maximum number of edges in an n-vertex oriented graph without a transitive triangle?

   (c) What is the maximum number of edges in an n-vertex *directed* graph without a cyclic triangle?

   (d) What is the maximum number of edges in an n-vertex *directed* graph without a transitive triangle?