# Exercises (recommended)

1. (a) Prove that every 2-coloring of $E(K_N)$ contains a monochromatic $N$-vertex tree.

   (b) Prove that for every $q \geqslant 2$, there exists some $\delta > 0$ such that the following holds. In any $q$-coloring of $E(K_N)$, one of the color classes contains *all* of the trees on $\delta N$ vertices.

2. (a) Let $G, H$ be graphs such that $H$ is connected. Prove that

   $$r(G, H) \geqslant (\chi(G) - 1)(|H| - 1) + 1,$$

   where $|H|$ denotes the number of vertices of $H$.

   (b) Let $\sigma(G)$ denote the minimum number of vertices that can appear in a color class among all proper $\chi(G)$-colorings of $G$. Strengthen the result above to

   $$r(G, H) \geqslant (\chi(G) - 1)(|H| - 1) + \sigma(G).$$

3. Let $\ell K_2$ denote the *matching graph*, consisting of $2\ell$ vertices and $\ell$ disjoint edges. Prove that $r(\ell K_2, K_k) = 2\ell + k - 2$ for all integers $\ell \geqslant 1, k \geqslant 2$.

4. (a) Let $k, \ell \geqslant 2$. Prove that in any sequence of $(k-1)(\ell-1)+1$ distinct real numbers, there is an increasing subsequence of length $k$ or a decreasing subsequence of length $\ell$.

   (b) Prove that the result in (a) is best possible, by finding a sequence of $(k-1)(\ell-1)$ distinct real numbers with no increasing subsequence of length $k$ and no decreasing subsequence of length $\ell$.

5. Prove that any sequence of (not necessarily distinct) real numbers of length $(k-1)^3+1$ contains a subsequence of length $k$ that is strictly increasing, strictly decreasing, or constant. Prove that this bound is best possible.

# Problems (optional)

✣1. (a) Prove that any infinite sequence of distinct real numbers contains an infinite subsequence that is (non-strictly) increasing or (non-strictly) decreasing.

   (b) Prove the Bolzano–Weierstrass theorem: every bounded sequence of real numbers has a convergent subsequence.

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$\star$ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

2. Let $v_1, \dots, v_N \in \mathbb{R}^d$ be vectors, and let $(v_i)_j$ denote the $j$th coordinate of $v_i$, for any $j \in [\![d]\!]$. Prove that if $N \geqslant (k-1)^{2^d} + 1$, then there is a *totally monotone* subsequence of length $k$; that is, there are $i_1 < \cdots < i_k$ such that for all $j \in [\![d]\!]$, we have $(v_{i_1})_j \geqslant \cdots \geqslant (v_{i_k})_j$ or $(v_{i_1})_j \leqslant \cdots \leqslant (v_{i_k})_j$.

⋆3. Prove[1] that for every $\Delta \geqslant 2$, there exists $C_\Delta > 0$ such that the following holds for every $n$. If $N \geqslant C_\Delta n$, then in any two-coloring of $E(K_N)$, one of the color classes contains *all* $n$-vertex graphs of maximum degree at most $\Delta$.

✣4. Prove that a graph has degeneracy at most $d$ if and only if its vertices can be ordered as $v_1, \dots, v_n$ such that $v_i$ has at most $d$ neighbors preceding it in the order.

*Remark:* This alternative definition is very useful when trying to prove things like the Burr–Erdős conjecture, as it suggests a good order in which to try to embed the vertices one by one.

⋆⋆5. (a) Formalize the proof sketch we saw in class, and prove that if $H$ is an $n$-vertex graph with maximum degree $\Delta$, then

$$r(H) \leqslant 2^{C\Delta(\log \Delta)^2} n,$$

where $C$ is an absolute constant.

⋆⋆(b) Improve this bound to

$$r(H) \leqslant 2^{C\Delta \log \Delta} n.$$

?(c) Improve this bound to

$$r(H) \leqslant 2^{C\Delta} n.$$

⋆6. Let $kK_3$ denote the graph that is the disjoint union of $k$ triangles. Prove[2] that for all $k \geqslant 2$, we have $r(kK_3) = 5k$.

7. Prove that $r(C_4; q) \leqslant q^2 + q + 2$ for all $q \geqslant 2$.

8. Erdős conjectured that if $\chi(H) = k$, then $r(H) \geqslant r(K_k)$.

(a) Prove this conjecture for $k \leqslant 3$.

⋆⋆(b) Find a counterexample to this conjecture for $k = 4$.

✣9. Prove that if $N$ is sufficiently large, then the following holds. Among any $N$ points in the plane, there are three of them that determine an angle greater than $179°$.

⋆⋆10. Construct[3], for every $k \geqslant 4$, a collection of $2^{k-2}$ points in $\mathbb{R}^2$, no three collinear, with no $k$ of them in convex position. Deduce that $\mathrm{Kl}(k) \geqslant 2^{k-2} + 1$.

[1] *Hint:* There is a black-box reduction to the statement of Theorem 15.12.
[2] *Hint*: Induct on $k$. The base case is super annoying, but the inductive step is nice.
[3] *Hint:* A solution for $k = 5$ is given on the next page; try to generalize it.