Exercises (recommended)

1. (a) Prove that for every forest F, there exists some integer N such that the following holds. In any coloring of $E(K_N)$, with an arbitrary number of colors, there is a monochroamtic or rainbow copy of F.

   (b) Prove that the result of part (a) is false for any graph H which is not a forest.

2. Let us say that a coloring of $E(K_k)$ is *semi-starry* if the vertices can be sorted as $v_1, \ldots, v_k$ such that all edges $v_i v_j$, where $j > i$, are of the same color. (The only difference from a starry coloring is that we do not require these colors to be distinct.)

   (a) Prove that if $N \geqslant (k-1)^2 + 1$, then any semi-starry coloring of $E(K_N)$ contains a monochromatic or starry K_k. Such a result was implicitly used in the proof of Theorem 16.3.

   (b) Prove that if $N \geqslant k^{4k}$, then any coloring of $E(K_N)$, with an arbitrary number of colors, contains a rainbow or a semi-starry K_k.

   ⋆(c) Show that, for some absolute constant $c > 0$, there exists a coloring of $E(K_N)$, where $N = k^{ck}$, with no rainbow or semi-starry K_k. Thus, the result of part (b) is best possible up to the constant factor in the exponent.

3. Usually, the canonical Ramsey theorem is stated in an ordered version. Here, we label $V(K_N)$ as $v_1, \ldots, v_N$, and we say that indices $i_1 < \cdots < i_k$ form a *left-starry* K_k if all edges from v_{i_j} to v_{i_ℓ} receive the same color, for all $\ell > j$, and these colors are distinct for different j. Similarly, it's *right-starry* if the same holds for all $\ell < j$.

   Prove that for every k, there exists some N such that any coloring of $E(K_N)$, with an arbitrary number of colors and with the fixed vertex labeling $v_1, \ldots, v_N$, there is a monochromatic, rainbow, left-starry, or right-starry K_k.

Problems (optional)

1. Prove the bipartite canonical Ramsey theorem, which states the following. For every $k \geqslant 2$, there exists some N such that in any coloring of $E(K_{N,N})$, with an arbitrary number of colors, there is a $K_{k,k}$ which is monochromatic, rainbow, or starry.

   (Here, a $K_{k,k}$ is *rainbow* if all k^2 edges receive different colors, and is *starry* if it is colored by exactly k distinct colors, each of whose color classes is a star $K_{1,k}$.)

✣ 2. Fekete's lemma is an important result in real analysis. It says that if a function $f : \mathbb{N} \to \mathbb{R}$ is *supermultiplicative*, meaning that $f(m+n) \geqslant f(m)f(n)$ for all m, n, then the limit $\lim_{n\to\infty} f(n)^{1/n}$ exists.

⋆ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

$\star$ (a) Prove Fekete's lemma (or take it as a given and move on).

(b) Prove that, for any fixed $k \geqslant 3$, the limit

$$\lim_{q\to\infty} r(k;q)^{1/q}$$

exists.

? (c) Prove that, for any fixed $q \geqslant 2$, the limit

$$\lim_{k\to\infty} r(k;q)^{1/k}$$

exists.

$\star$ 3. There is also a canonical Ramsey theorem for hypergraphs.

(a) Find a list of colorings of $K_N^{(3)}$ that are canonical, in the sense that every subset of vertices is colored in the same way. Monochromatic and rainbow are obvious examples, but what are the correct hypergraph notions of starry?

$\star$ (b) Prove the canonical Ramsey theorem for 3-uniform hypergraphs: for every k, there exists some N such that in any coloring of $E(K_N^{(3)})$, with an arbitrary number of colors, there is a copy of $K_k^{(3)}$ that is colored according to one of the canonical colorings in the list you found.

$\star\star$ (c) Extend these results to t-uniform hypergraphs.