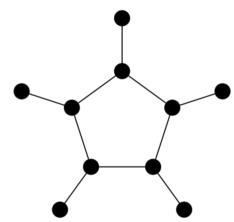
Exercises (recommended)

1. Let P_k denote the path graph with k vertices and $k-1$ edges.
   (a) Prove that, for any $q \geqslant 2$, the star $K_{1,q+1}$ is q-color Ramsey for P_3.
   (b) Prove that the following graph, obtained by adding a leaf to each vertex of C_5, is 2-color Ramsey for P_4.



   (c) Give an example of a graph that is 2-color Ramsey for P_5.

2. Prove that for every $n, q \geqslant 2$, there exists some N such that $K_{N,N}$ is q-color Ramsey for $K_{n,n}$.

3. A graph G is *minimally Ramsey* for H if G is Ramsey for H, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H. H is called *Ramsey finite* if there are only a finite number of minimally Ramsey graphs for H, and *Ramsey infinite* otherwise.
   (a) Let $G = K_3 * C_\ell$, where $\ell \geqslant 3$ is odd. Prove that G is minimally Ramsey for K_3. Conclude that K_3 is Ramsey infinite.
   (b) Determine the set of Ramsey minimal graphs for $K_{1,2}$.
   $\star$ (c) Prove that $K_{1,k}$ is Ramsey finite if and only if k is odd.

4. A graph G is called *q-minimally Ramsey* for a graph H if G is Ramsey for H in q colors, but any proper subgraph $G' \subsetneq G$ is not Ramsey for H in q colors.
   (a) Prove that if G is q-minimally Ramsey for H, then every edge of G lies in at least q copies of H.
   (b) Prove that if G is q-minimally Ramsey for H, then G has at least $q^{e(H)-1}$ copies of H.
   (c) Prove Proposition 17.9 from the notes.

$\star$ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

Problems (optional)

1. (a) Prove that if G is Ramsey for K_k, then $\chi(G) \geqslant r(k)$.
   (b) Prove that if G is Ramsey for K_k, then $e(G) \geqslant \binom{r(k)}{2}$. That is, the complete graph $K_{r(k)}$ has the fewest number of edges among all graphs Ramsey for K_k.
   (c) Find an example of a graph H for which there is a graph G which is Ramsey for it, but G has fewer edges than $\binom{r(H)}{2}$. That is, it was really important that we took $H = K_k$ above.

⋆2. Recall the definitions from exercise 3. In this problem, you'll prove that every tree which is not a star is Ramsey infinite.
   (a) Prove that if T is an n-vertex tree, then every graph G of chromatic number at least $n^2 + 1$ is Ramsey for G.
   (b) Prove that if T is a tree which is not a star, then for any forest F, F is not Ramsey for T.
   ⋆(c) Prove[1] that for every $k, g \geqslant 3$, there is a graph G with chromatic number at least k and girth at least g (the *girth* of G is the length of the shortest cycle in G).
   (d) Using the previous parts, prove that for every tree T which is not a star, and for every integer g, there is a graph G on at least g vertices which is minimally Ramsey for T. Conclude that T is Ramsey infinite.

3. (a) Prove that for any $\ell \geqslant 4$, the cycle C_ℓ is a subgraph of a triangle tree (and hence Ramsey obligatory for K_3).
   (b) Prove that K_4 is not a subgraph of any triangle tree.

✣4. (a) Prove that if G is Ramsey for H and H is Ramsey for F, then G is 4-color Ramsey for F.
   (b) Find an example[2] of G, H, F as above for which G is *not* 5-color Ramsey for F.

5. Let G be a graph. Recall that the *s-blowup* of G, denoted $G[s]$, is the graph obtained by replacing each vertex of G by s vertices, and replacing each edge of G by a complete bipartite graph $K_{s,s}$.
   (a) Prove that for every $s \geqslant 2$, there exists some $N = N(s)$ such that $K_6[N]$ is Ramsey for $K_3[s]$.
   (b) Prove that $N(s) > 2^s$ for all $s \geqslant 4$.
   ⋆(c) Prove the following generalization of part (a): For every graph H and every integer s, if G is Ramsey for H, then there exists N such that $G[N]$ is Ramsey for $H[s]$.

[1] Hint: Consider a random n-vertex graph where each edge is included with probability $p = n^{-1/(2g)}$.
[2] *Hint:* $H = K_3$.

✣⋆⋆6. Prove[3] Theorem 17.3 from Theorem 17.5. You should in fact assume the following strengthening of Theorem 17.5; in the same setup as in the theorem statement, we have that

$$\Pr(G \text{ is Ramsey for } H \text{ in } q \text{ colors}) \begin{cases} \geqslant 1 - e^{-cpN^2} & \text{if } p \geqslant CN^{-1/m_2(H)}, \\ \leqslant e^{-cpN^2} & \text{if } p \leqslant cN^{-1/m_2(H)}. \end{cases}$$

[3] *Hint:* Use Harris's inequality/FKG inequality (and look it up if you've never heard of it).