Exercises (recommended)

1. Let $\mathcal{H}$ be a collection of graphs. We say that G is $\mathcal{H}$-*free* if G has no copy of any $H \in \mathcal{H}$, and we define

$$\mathrm{ex}(n, \mathcal{H}) = \max\{e(G) : G \text{ is an } n\text{-vertex } \mathcal{H}\text{-free graph}\}.$$

Assuming the Erdős–Stone–Simonovits theorem, prove that

$$\mathrm{ex}(n, \mathcal{H}) = \left(1 - \frac{1}{\chi(\mathcal{H}) - 1} + o(1)\right)\binom{n}{2},$$

where $\chi(\mathcal{H}) \coloneqq \min\{\chi(H) : H \in \mathcal{H}\}$.

2. By carefully analyzing the proof we saw in class, prove that $\mathrm{ex}(n, C_4) \leqslant \frac{n(\sqrt{4n-3}+1)}{4}$.

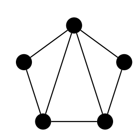
Hint: Start with the case that G is d-regular, where $d \geqslant \frac{\sqrt{4n-3}+1}{2}$.

3. Today we proved that for any graph H,

$$\mathrm{ex}(n, H) \geqslant t_{\chi(H)-1}(n), \qquad (*)$$

which in particular implies the lower bound in the Erdős–Stone–Simonovits theorem. In this problem, you'll see examples of graphs where inequality $(*)$ is not best possible, i.e. where the Turán graph $T_{\chi(H)-1}(n)$ has strictly fewer edges than $\mathrm{ex}(n, H)$.

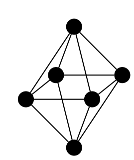
   (a) Let H be the graph



   Verify that $\chi(H) = 3$, so that inequality $(*)$ implies $\mathrm{ex}(n, H) \geqslant t_2(n) = \lfloor n^2/4 \rfloor$.

   (b) Add some edges to the Turán graph $T_2(n)$ to prove that $\mathrm{ex}(n, H) \geqslant \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{n}{4} \rfloor$.

   ⋆ (c) Let O_3 be the graph corresponding to the octahedron, namely the graph



   Verify that $\chi(O_3) = 3$. Add edges to $T_2(n)$ to prove that

$$\mathrm{ex}(n, O_3) \geqslant \left\lfloor \frac{n^2}{4} \right\rfloor + cn^{3/2},$$

⋆ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

for some absolute constant $c > 0$.

Hint: You may assume the fact that I stated but didn't prove in class, namely that $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$ (i.e. that we have a matching lower bound to the upper bound we proved).

(d) Why don't these examples violate the Erdős–Stone–Simonovits theorem?

4. Provide an alternative proof of Turán's theorem using a technique called *Zykov symmetrization*. Let G be a K_r-free n-vertex graph.

   (a) Pick two non-adjacent vertices $x, y \in V(G)$, and assume without loss of generality that $\deg(x) \geqslant \deg(y)$. Replace y with a *clone* of x, i.e. another vertex x' with the same neighborhood as x.

   (b) Repeat step (a) over and over until doing so no longer changes the graph (and prove that this must eventually happen).

   (c) Prove that the resulting graph when you get stuck is complete $(r-1)$-partite.

   (d) Conclude that $e(G) \leqslant t_{r-1}(n)$, with equality if and only if $G \cong T_{r-1}(n)$.

Problems (optional)

✣⋆1. Suppose $p_1, \ldots, p_n \in \mathbb{R}^2$ are n points in the plane. Prove that the number of *unit distances* among them (i.e. pairs $\{p_i, p_j\}$ with $\|p_i - p_j\| = 1$) is at most $O(n^{3/2})$.

Can you prove a stronger upper bound, or find a matching lower bound?

⋆2. Let G be an n-vertex triangle-free graph.

   (a) Suppose every vertex of G has degree greater than $2n/5$. Prove that G is bipartite.

   (b) Show that $2/5$ is the optimal constant in this theorem, that is, that for every n, there exists a non-bipartite triangle-free graph with minimum degree $\lfloor 2n/5 \rfloor$.

   ⋆⋆(c) Can you find generalizations of parts (a) and (b) for K_r-free graphs, $r > 3$?

✣3. In this problem you will prove Jensen's inequality in full generality.

   (a) A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

   Prove that if f is twice-differentiable and satisfies $f'' \geqslant 0$, then f is convex.

   (b) Suppose f is convex. Let $x_1, \ldots, x_n \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\lambda_1 + \cdots + \lambda_n = 1$. Prove that

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geqslant f\left(\sum_{i=1}^{n} \lambda_i x_i\right)$$

   by induction on n. This is the general form of Jensen's inequality.

   (c) Prove that $f(x) = \binom{x}{r}$ is convex on the interval $[r, \infty)$ using part (a), and conclude the version of Jensen's inequality that I stated in class from part (b).