Exercises (recommended)

1. (a) Suppose that T is a tree with $t+1$ vertices, and G is a graph with minimum degree at least t. Prove that G contains a copy of T.
   Hint: Induction on t.

   (b) Let G be an n-vertex graph with m edges. Prove that there is a subgraph $G' \subseteq G$ with minimum degree strictly greater than m/n.
   Hint: Repeatedly delete vertices of degree $\leqslant m/n$.

   (c) Using parts (a) and (b), prove that if T is a tree with $t+1$ vertices, then
   $$\mathrm{ex}(n,T) < (t-1)n.$$

   (d) Prove that if n is divisible by t, then
   $$\mathrm{ex}(n,T) \geqslant \frac{(t-1)n}{2}.$$

   ? (e) Erdős and Sós conjectured that the lower bound in part (d) is best possible, i.e. that
   $$\mathrm{ex}(n,T) = \left\lfloor \frac{(t-1)n}{2} \right\rfloor$$
   for all $(t+1)$-vertex trees T. Can you prove or disprove this conjecture?

2. Let $K_{1,r}$ denote the star with r leaves. Determine $\mathrm{ex}(n,K_{1,r})$ for all n and r. Is your answer consistent with the Erdős–Sós conjecture from exercise 1? Is it consistent with the Kővári–Sós–Turán theorem we proved in class?

3. Recall that we defined
   $$m_2(H) = \max_{F \subseteq H} \frac{e(F)-1}{v(F)-2},$$
   and stated in class that $\mathrm{ex}(n,H) \geqslant \Omega(n^{2-1/m_2(H)})$ for all bipartite H.

   (a) Compute $m_2(C_{2\ell})$ for each $\ell \geqslant 2$. What lower bound on $\mathrm{ex}(n,C_{2\ell})$ do you get?

   (b) Compute $m_2(K_{s,t})$ for all $t \geqslant s \geqslant 2$. How does the resulting lower bound compare to the others we've discussed?

   (c) Compute $m_2(T)$ for any tree T. How does the resulting lower bound relate to exercise 1?

   ⋆ (d) Pick your favorite bipartite graph, and compute the lower and upper bounds coming from $m_2(H)$ and from finding H as a subgraph of $K_{s,t}$, respectively. Can you improve either of these bounds?

4. Using previous homework problems, prove the following fact. A graph H is a forest if and only if $\mathrm{ex}(n,H) \leqslant O(n)$.

⋆ means that a problem is hard.
? means that a problem is open.
✣ means that a problem is on a topic beyond the scope of the course.

Problems (optional)

$\star$ 1. In this problem, you'll prove that $\mathrm{ex}(n,H) \geqslant \Omega(n^{2-1/m_2(H)})$. This problem requires some background in probability, specifically linearity of expectation.

(a) Let $p \in [0,1]$, and let G be a *random* n-vertex graph obtained by making every pair of vertices adjacent with probability p, independently over all choices. Prove that the expected number of edges in G is $p\binom{n}{2}$.

(b) Prove that for any fixed graph H, the expected number of copies of H in G is at most $p^{e(H)}n^{v(H)}$.

(c) Suppose that H is *2-balanced*, meaning that in the definition of $m_2(H)$, the maximizing subgraph F is H itself. Let X denote the random variable defined as the number of edges of G minus the number of copies of H in G. Prove that if $p = cn^{-1/m_2(H)}$, for some appropriate constant $c > 0$, then $\mathbb{E}[X] \geqslant \Omega(n^{2-1/m_2(H)})$.

(d) Prove that if H is 2-balanced, then $\mathrm{ex}(n,H) \geqslant \Omega(n^{2-1/m_2(H)})$.

(e) Prove that the same conclusion holds even if H is not 2-balanced.

$\star$ 2. In this problem you'll prove the Erdős–Sós conjecture in the special case that T is a path. By the *length* of a path, we mean the number of vertices it has.

$\star\star$ (a) Let G be an n-vertex connected graph with minimum degree $\delta(G)$. Prove that G contains a path of length at least $\min\{n, 2\delta(G)+1\}$.
Hint: Consider a longest path in G, and try to extend it.

(b) Let P_{t+1} denote the path of length $t+1$. Prove that

$$\mathrm{ex}(n, P_{t+1}) \leqslant \left\lceil \frac{(t-1)n}{2} \right\rceil.$$

Hint: Induction on n.

$\star$ (c) Can you characterize the extremal graphs, i.e. the P_{t+1}-free graphs with the maximum number of edges?

3. Provide an alternative proof of Turán's theorem using induction on r. Let G be a K_r-free n-vertex graph.

(a) Let v be a vertex of maximum degree in G. Let A be the set of neighbors of v, and let $B = V(G) \setminus A$.

(b) Form a new graph H by deleting all edges inside B, and adding in all missing edges between A and B. Prove that $e(H) \geqslant e(G)$.

(c) Apply the inductive hypothesis (Turán's theorem for $r-1$) to the induced subgraph on A. Conclude that Turán's theorem holds for r.